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Stochastic analysis on infinite dimensional spaces

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Contents

Preface

The U.S.-Japan Bilateral Seminar "*Stochastic Analysis on Infinite Dimensional Spaces*" was held at Louisiana State University, January 4-8, 1994. The seminar covered the following topics:

(1) Stochastic analysis related to Lie groups.
(2) Stochastic partial differential equations.
(3) Stochastic flows and analysis on Wiener functionals.
(4) Large deviations.
(5) White noise calculus.
(6) Stable laws.

This volume is the collection of all lectures delivered during this seminar. We would like to thank all contributors for their participation in this seminar and for their effort to prepare the manuscript in a timely manner.

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The non-equivalence of Dirichlet forms on path spaces

1. Introduction

To each finite dimensional Riemannian manifold M with covariant derivative ∇ is an associated Dirichlet form $(\mathcal{E}^\nabla)$ on the path space $W(M)$ of M. If $M = \mathbb{R}^d$ and ∇ is the Levi-Civita covariant derivative on $\mathbb{R}^d$, then $Q \equiv \mathcal{E}^\nabla$ is the "usual" Dirichlet form on the classical Wiener space. These lecture notes will discuss the relationship (or lack of) between $\mathcal{E}^\nabla$ and Q as (M, ∇) varies.

2. Smooth Preliminaries

Let $(M^d, \langle\cdot,\cdot\rangle, \nabla, o)$ be given, where M is a compact connected manifold (without boundary) of dimension d, $\langle\cdot,\cdot\rangle$ is a Riemannian metric on M, ∇ is a $\langle\cdot,\cdot\rangle$-compatible covariant derivative, and o is a fixed base point in M. Let $T = T^\nabla$ and $R = R^\nabla$, denote the torsion and curvature of ∇ respectively. We denote parallel translation up to time "s" along a smooth path $\sigma : [0,1] \to M$ by $P_s(\sigma) = P_s^\nabla(\sigma)$.

Standing Assumption: The covariant derivative (∇) is assumed to be *Torsion Skew Symmetric* or *TSS* for short. That is to say, if $T = T^\nabla$ is the torsion tensor of ∇, then $\langle T(X,Y), Y\rangle \equiv 0$ for all vector fields X and Y on M. (With the TSS condition, the Laplacian on functions $(\Delta f = \mathrm{tr}(\nabla \mathrm{grad} f))$ associated to ∇ is the same as the usual Levi-Civita Laplacian.)

2.1. Examples of (M, ∇)

Example 2.1 Let $M = SO(n)$ – the $n \times n$ real orthogonal matrices g with $\det(g) = 1$. (In this case $d = n(n-1)/2$.) Take $o = I$, $\langle A, B\rangle \equiv \mathrm{tr}(A^t B)$ for $A, B \in T_g G$ — the set of $n \times n$ real matrices (A) such that $g^{-1}A$ is skew symmetric. There are three natural covariant derivatives on G : namely the *left* (∇^L), *right* (∇^R), and *Levi-Civita* (∇^{Levi}) covariant derivative. The left and right covariant derivatives may be described by describing how they act on a vector field $A(t) \in T_{g(t)}G$ along a curve $g(t) \in G$. The formulas are:

$$\frac{\nabla^L A(t)}{dt} = g(t)\frac{d(g(t)^{-1}A(t))}{dt}$$

and

$$\frac{\nabla^R A(t)}{dt} = \frac{d(A(t)g(t)^{-1})}{dt} g(t).$$

The Levi-Civita covariant derivative is the average ($\nabla^{Levi} = (\nabla^L + \nabla^R)/2$) of the left and the right covariant derivatives.

With these definitions we have the following table

∇	$P_s^\nabla(\sigma)A$	$R^\nabla < A, B > C$	$T^\nabla < A, B >$
∇^L	$\sigma(s)A$	0	$-g[g^{-1}A, g^{-1}B]$
∇^R	$A\sigma(s)$	0	$[Ag^{-1}, Bg^{-1}]g$
∇^{Levi}	$***$	$g[[g^{-1}A, g^{-1}B], g^{-1}C]/4$	0

where $A, B, C \in T_g G$ and $[A, B] \equiv AB - BA$ when A and B are square matrices. The entry for parallel translation for the Levi-Civita covariant derivative is left blank, since no explicit formula for $P_s^\nabla(g)$ can in general be given when ∇ has nonzero curvature.

Example 2.2 Let M be an oriented hyper-surface in $\mathbb{R}^d$ and $\langle\cdot,\cdot\rangle$ be the usual inner product on $\mathbb{R}^d$. Let $N : M \to S^{d-1} \subset \mathbb{R}^d$ be a smoothly varying unit normal vector on M. If $X(t)$ is a smoothly varying tangent vector to M along a smooth curve $\sigma(t)$ in M, define

$$\nabla X(t)/dt = Q(\sigma(t))dX(t)/dt = dX(t)/dt + \{X(t) \cdot dN(\sigma(t))/dt\}N(\sigma(t)),$$

where $Q(\sigma(t))$ denotes orthogonal projection onto the tangent plane to M at $\sigma(t)$. This covariant derivative is the Levi-Civita covariant derivative on M. The torsion tensor (T^∇) is zero and the curvature tensor ($R \equiv R^\nabla$) is given by

$$R\langle v, w\rangle z = \langle dN\langle w\rangle, z\rangle dN\langle v\rangle - \langle dN\langle v\rangle, z\rangle dN\langle w\rangle,$$

where v, w, z are tangent vectors to M at (say) $x \in M$ and

$$dN\langle v\rangle \equiv \frac{d}{dt}\Big|_0 N(\sigma(t)),$$

where $\sigma : (-1, 1) \to M$ is any smooth curve in M such that $\sigma(0) = x$ and $\dot\sigma(0) = v$.

3. Stochastic Preliminaries

Let

$$W(M) \equiv \{\sigma \in C([0,1], M) | \sigma(0) = o\},$$

$$W \equiv \{\omega \in C([0,1], T_o M) | \omega(0) = 0 \in T_o M\},$$

(i.e. $W \equiv W(T_o M)$), μ be Wiener measure on W, and ν be Wiener measure on $W(M)$. As usual $H \subset W$ will denote the Cameron-Martin subspace of W consisting of those paths $h \in W$ such that $(h,h)_H \equiv \int_0^1 |h'(s)|^2 ds < \infty$, where $|h'(s)|^2 \equiv \langle h'(s), h'(s)\rangle$. Let $\{P_s^\nabla\}_{s\in[0,1]}$ denote a fixed version of the stochastic parallel transport process on $W(M)$. So

$$P_s^\nabla(\sigma) : T_o M \to T_{\sigma(s)} M$$

is an isometry for all $s \in [0, 1]$.

Example 3.1 Take $M = G = SO(n)$ and ∇ to be either ∇^L or ∇^R, then $P_s^L(\sigma)A \equiv P_s^{\nabla^L}(\sigma)A = \sigma(s)A$ and $P_s^R(\sigma)A \equiv P_s^{\nabla^R}(\sigma)A = A\sigma(s)$ respectively.

Example 3.2 If M is an embedded hypersurface of $\mathbb{R}^d$, then $P_s^\nabla(\sigma)$ may be thought of as a version of the $d \times d$-matrix valued solution to the stochastic differential equation

$$dP_s^\nabla(\sigma) + N(\sigma(s))\{\delta(N(\sigma(s))\}^t P_s^\nabla(\sigma) = 0 \text{ with } P_0^\nabla(\sigma) = I, \tag{3.1}$$

where δ denotes the Stratonovich differential.

3.1. Stochastic development

There is a well known measure theoretic isomorphism (Ψ^∇) between (W, μ) and $(W(M), \nu)$. The map $\Psi \equiv \Psi^\nabla$ is defined uniquely up to μ-equivalence as the solution to the stochastic (functional) differential equation:

$$\delta\Psi_s(\omega) = P_s^\nabla(\Psi_\cdot(\omega))\delta\omega \text{ with } \Psi_0(\omega) = o, \tag{3.2}$$

where δ is used to denote the Stratonovich differential and ω is a Wiener path in W, see Eells and Elworthy [8] and Malliavin [16]. The measure theoretic inverse to Ψ^∇ will be denoted by b^∇. Notice that b^∇ is an $\mathbb{R}^d$-valued Brownian motion defined on the probability space $(W(M), \nu)$.

It is well known that Ψ carries the measure μ to ν and that Ψ is invertible up to equivalence. However, as pointed out by Malliavin [17, 18], the map Ψ does not preserve the natural "Riemannian metrics" on W and $W(M)$ except in the case that (M, ∇) itself has trivial geometry.

Example 3.3 Let $M = G = SO(n)$, $\nabla = \nabla^L$ or ∇^R, and $\Psi^L \equiv \Psi^{\nabla^L}$ or $\Psi^R \equiv \Psi^{\nabla^R}$ respectively. Then the functions Ψ^L and Ψ^R are solutions to the stochastic differential equations

$$\delta\Psi_s^L(\omega) = \Psi_s^L(\omega)\delta\omega(s) \text{ with } \Psi_0^L(\omega) = I$$

and

$$\delta\Psi_s^R(\omega) = \delta\omega(s)\Psi_s^R(\omega) \text{ with } \Psi_0^R(\omega) = I$$

respectively.

Remark 3.4 For embedded surfaces, bounding an open convex subset of $\mathbb{R}^3$, equipped with the induced Riemannian structure, the map Ψ has the interpretation of transferring the path ω in $\mathbb{R}^2$ to a path on the surface by "rolling" the surface along ω without slipping.

3.2. Flows, quasi-invariance, and integration by parts

The space $W(M)$ is to be thought of as an infinite dimensional manifold. To understand the notion of a tangent vector to $W(M)$ at $\sigma \in W(M)$, consider a differentiable curve $(t \in (-1,1) \to f(t,\cdot) \in W(M))$ such that $f(0,\cdot) = \sigma(\cdot)$. The derivative $X(s) \equiv \frac{d}{dt}|_0 f(t,s)$ of such a curve in $W(M)$ is a vector-field along σ, i.e. $X \in C([0,1], TM)$ such that $X(s) \in T_{\sigma(s)}M$ for all $s \in [0,1]$. So it is reasonable to say that a vector-field X along $\sigma \in W(M)$ is in the tangent space $(T_\sigma W(M))$ to $W(M)$ at σ.

Example 3.5 For each $h \in H$ let X^h be the vector-field on $W(M)$ defined by

$$X^h(\sigma)(s) = P_s^\nabla(\sigma)h(s) \ \forall s \in [0,1]. \tag{3.3}$$

Notice that $X^h(\sigma) \in T_\sigma W(M)$.

Theorem 3.6 **(Quasi-invariant Flow)** *Let $h \in H$ and X^h be the vector-field in (3.3). Then X^h admits a flow $(e^{tX^h})_{t\in\mathbb{R}}$ on $W(M)$ which leaves Wiener measure (ν) quasi-invariant. That is for each $t \in \mathbb{R}$, $\nu \circ e^{tX^h}$ and ν are mutually absolutely continuous relative to each other.*

This theorem was first proved in Driver [4] (see Theorem 8.5.) under the hypothesis that $h \in H \cap C^1$. The extension to the $h \in H$ was proved by Elton Hsu in [13, 14]. In the case that the manifold $M = G$ is a Lie group or a homogeneous space, the quasi-invariance of left and right multiplication by finite energy paths has been extensively studied, see [1, 10, 19, 20, 21].

Definition 3.7 A *smooth cylinder function* on $W(M)$ is a function f of the form

$$f(\sigma) = F(\sigma(s_1), \sigma(s_2), \ldots, \sigma(s_n)),$$

where $F: M^n \to \mathbb{R}$ is smooth and $\{s_1, s_2, \ldots, s_n\} \subset [0,1]$. Let $\mathcal{F}$ denote the class of smooth cylinder functions on $W(M)$.

Theorem 3.8 **(Integration by Parts)** *Let $f \in \mathcal{F}$ and $X \equiv X^h$ be as in (3.3). Define the action of X on f by the formula:*

$$Xf = L^2 - \lim_{t\to 0}(f \circ e^{tX} - f)/t.$$

Then the L^2-adjoint X^* of X is densely defined. Furthermore, X^* acting on cylinder functions is given by:

$$X^* = -X + z^h,$$

where z^h is a certain function on $W(M)$ constructed from the curvature tensor, the torsion tensor, and the parallel translation operator P_s^∇. (See Theorem 9.1 of Driver [4] for the explicit formula for z^h.)

This integration by parts theorem is a Corollary of the previous theorem, see [4]. It is also possible to prove and extend Theorem 3.8 by using ideas of Bismut [2]; see Leandre [15], and Fang and Malliavin [9].

Definition 3.9 The *gradient operator* $D^\nabla : \mathcal{F} \to \mathrm{L}^2(\nu, H)$ is defined by

$$D^\nabla f \equiv \sum_{h\in S} X^h f \otimes h, \quad f \in \mathcal{F},$$

where $S \subset H$ is an orthonormal basis for H.

Using Theorem 3.8, it is easy to see that D^∇ is a closable operator. We will continue to denote the closure by D^∇. Associated to D^∇ is the closed quadratic form $\mathcal{E}^\nabla(\cdot,\cdot)$ on $\mathrm{L}^2(\nu)$ defined by

$$\mathcal{E}^\nabla(f,g) \equiv \int_{W(M)} (D^\nabla f, D^\nabla g)d\nu \ \forall f,g \in \mathcal{D}(D^\nabla), \tag{3.4}$$

where $\mathcal{D}(D^\nabla)$ is the domain of D^∇. The following theorem is from Driver and Röckner [7].

Theorem 3.10 *The form $\mathcal{E}^\nabla$ is a local quasi-regular Dirichlet form and hence there exists an associated diffusion process on $W(M)$. (This process is the Ornstein Uhlenbeck process when $M = \mathbb{R}^d$.)*

4. Non-Comparability of the Pull-Back Form

Using the stochastic development map Ψ^∇ and its inverse b^∇ it is possible to "pull-back" Dirichlet forms on $W(M)$ to Dirichlet forms on W. It is natural to compare these pulled back forms with the "usual" Dirichlet form Q on W.

Definition 4.1 The *usual Dirichlet* form on W is the closed symmetric quadratic form Q on $L^2(W,\mu)$ determined by

$$Q(F,F) = \int_W (DF, DF)_H d\mu, \tag{4.1}$$

where $D : L^2(W,\mu) \to L^2(W,H)$ is the closed operator determined on cylinder functions ($F : W \to \mathbb{R}$) by

$$DF \equiv \sum_{h\in S} \partial_h F \otimes h.$$

Here $\partial_h F(\omega) = L^2 - \lim_{t\to 0}(F(\omega+th) - F(\omega)/t)$ and $S \subset H$ is an orthonormal basis of H.

Definition 4.2 Let $\mathcal{E}^\nabla$ be as in (3.4), the *pull-back* of $\mathcal{E}^\nabla$ by the development map Ψ^∇ is the symmetric quadratic form Q^∇ on $L^2(W,\mu)$ determined by:

$$Q^\nabla(F,F) \equiv \mathcal{E}^\nabla(F\circ b^\nabla, F\circ b^\nabla)\ \forall F \in \mathcal{D}(Q^\nabla),$$

where

$$\mathcal{D}(Q^\nabla) \equiv \{F \in L^2(W,\mu) : F\circ b^\nabla \in \mathcal{D}(\mathcal{E}^\nabla) = \mathcal{D}(D^\nabla)\}.$$

(Recall that $b^\nabla \equiv (\Psi^\nabla)^{-1}$.)

Definition 4.3 Let Q and Q' be two closed symmetric non-negative quadratic forms on $L^2(W,\mu)$. Q and Q' are said to be *comparable* if $\mathcal{D}(Q) = \mathcal{D}(Q')$ and there is a constant $0 < C < \infty$ such that

$$C^{-1}Q(f,f) \le Q'(f,f) \le CQ(f,f)\ \forall f \in \mathcal{D}(Q) = \mathcal{D}(Q').$$

Question 1 Are Q and Q^∇ comparable?

Question 2 Given two covariant derivative $\nabla^{(1)}$ and $\nabla^{(2)}$ on M, are the Dirichlet forms $\mathcal{E}^1 \equiv \mathcal{E}^{\nabla^{(1)}}$ and $\mathcal{E}^2 \equiv \mathcal{E}^{\nabla^{(2)}}$ comparable?

We have the following negative answer in the case that $M = G = SO(n)$, $\nabla^{(1)} = \nabla^L$ and $\nabla^{(2)} = \nabla^R$.

Theorem 4.4 *Let $M = SO(n)$ (more generally a compact Lie group), $Q^L \equiv Q^{\nabla^L}$, $Q^R \equiv Q^{\nabla^R}$, $\mathcal{E}^L \equiv \mathcal{E}^{\nabla^L}$, and $\mathcal{E}^R \equiv \mathcal{E}^{\nabla^R}$. Then*

1. *$\sup_F Q(F)/Q^L(F) = \infty$ and $\sup_F Q^L(F)/Q(F) = \infty$.*
2. *$\sup_F Q(F)/Q^R(F) = \infty$ and $\sup_F Q^R(F)/Q(F) = \infty$.*
3. *$\sup_f \mathcal{E}^R(f)/\mathcal{E}^L(f) = \infty$ and $\sup_f \mathcal{E}^L(f)/\mathcal{E}^R(f) = \infty$.*

In the above expressions, the supremum is taken over the intersection of the domain of the quadratic form in the numerator with that in the denominator.

For a proof and a more detailed statement of the above theorem, see [6]. I conjecture that analogues of the above result hold for arbitrary Riemannian manifolds (M) with non-trivial geometry.

5. The (Non) Equivalence of the Forms

Because of the above theorem, it is natural to look for a weaker notion for the equivalence of quadratic forms.

Definition 5.1 Let (W,μ) and (W,μ') be two probability spaces and let Q and Q' be closed symmetric non-negative forms on $L^2(\mu)$ and $L^2(\mu')$, respectively. Then Q and Q' are said to be *equivalent* if there exists measurable maps $\phi : W \to W'$ and $\Psi : W' \to W$ such that:

1. $\Psi\circ\phi = id_W$ μ-a.s. and $\phi\circ\Psi = id_{W'}$ μ'-a.s.
2. $\mathcal{D}(Q) = \{f \in L^2(\mu) | f\circ\Psi \in \mathcal{D}(Q')\}$ and $\mathcal{D}(Q') = \{f \in L^2(\mu') | f\circ\phi \in \mathcal{D}(Q)\}$.
3. There is a constant $0 < C < \infty$ such that

$$C^{-1}Q'(f\circ\Psi, f\circ\Psi) \le Q(f,f) \le CQ'(f\circ\Psi, f\circ\Psi)\ \forall f \in \mathcal{D}(Q)$$

and

$$C^{-1}Q(f\circ\phi, f\circ\phi) \le Q'(f,f) \le CQ(f\circ\phi, f\circ\phi)\ \forall f \in \mathcal{D}(Q').$$

We now modify questions 1 and 2 above.

Question 1′ Is Q equivalent to $\mathcal{E}^\nabla$ or, equivalently, is Q equivalent to Q^∇?

Question 2′ Let $\nabla^{(1)}$ and $\nabla^{(2)}$ be two covariant derivatives on M. Are the Dirichlet forms $\mathcal{E}^1 \equiv \mathcal{E}^{\nabla^{(1)}}$ and $\mathcal{E}^2 \equiv \mathcal{E}^{\nabla^{(2)}}$ equivalent?

The answer to both questions for $M = SO(n)$ (or more generally a compact Lie group) with $\nabla = \nabla^L$ or $\nabla = \nabla^R$, $\nabla^{(1)} = \nabla^L$ and $\nabla^{(2)} = \nabla^R$ is now yes. The following theorem is essentially Theorem 3.14 in Gross [12].

Theorem 5.2 *Suppose that $M = G$ is a compact Lie group (ex. $G = SO(n)$) $Q^L \equiv Q^{\nabla^L}$, $Q^R \equiv Q^{\nabla^R}$, $\mathcal{E}^L \equiv \mathcal{E}^{\nabla^L}$, and $\mathcal{E}^R \equiv \mathcal{E}^{\nabla^R}$. Then Q, Q^L, Q^R, $\mathcal{E}^L$, and $\mathcal{E}^R$ are all equivalent. Moreover:*

$$\mathcal{E}^L(F \circ b^R, F \circ b^R) = Q(F,F) = \mathcal{E}^R(F \circ b^L, F \circ b^L),$$

where $b^L \equiv b^{\nabla^L}$ and $b^R \equiv b^{\nabla^R}$.

Unfortunately, as we will discuss below, the results of Theorem 4.4 seem to be essentially restricted to the case that G is a compact Lie group. To investigate this lack of equivalence of forms, let us attempt to find an invertible measurable map $b : W(M) \to W$ such that (i) $b_*\nu \equiv \nu \circ b^{-1}$ is equivalent to μ, (ii) b is adapted, and (iii) $Q(f,f)$ is comparable to $\mathcal{E}^\nabla(f \circ b, f \circ b)$ for all $f \in \mathcal{D}(Q)$. (Note: it would be preferable to drop condition (ii) above, but at the present time I do not know how to handle the anticipatory case.) With the above assumptions, $b = \phi \circ b^\nabla$ for some adapted map $\phi : W \to W$ such that $\phi_*\mu$ and μ are equivalent. The following "structure" theorem is proved in Driver [5], see Theorem 2.1.

Theorem 5.3 (Structure Theorem) *Let $\phi : W \to W$ be an adapted map such that $\phi_*\mu$ is equivalent to μ and there is an adapted map $\phi^{-1} : W \to W$ such that $\phi \circ \phi^{-1}$ and $\phi^{-1} \circ \phi$ are both equal to the identity map μ-a.s. Then there exists an $O(d) \times \mathbb{R}^d$-valued predictable process $(\bar{O}, \bar{\alpha})$ on W such that*

$$\phi(\omega) = \int \bar{O}(\omega) d\omega + \int \bar{\alpha}(\omega) ds, \tag{5.1}$$

and $\int_0^1 |\bar{\alpha}_{s'}|^2 ds' < \infty$ μ-a.s.

Because of this theorem, we learn that $b = \phi \circ b^\nabla$ may be written as

$$b = \int O db^\nabla + \int \alpha ds, \tag{5.2}$$

where $O \equiv \bar{O} \circ b^\nabla$ and $\alpha \equiv \bar{\alpha} \circ b^\nabla$. So our goal is to choose a predictable $O(d) \times \mathbb{R}^d$-valued process (O, α) on $W(M)$ such that $Q(f,f)$ is comparable to $\mathcal{E}^\nabla(f \circ b, f \circ b)$.

Notation 5.4 For $a, b \in T_oM$ and an isometry $u : T_oM \to T_mM$, let

$$\Omega_u\langle a, b\rangle \equiv u^{-1} R^\nabla \langle ua, ub\rangle u \in End(T_oM)$$

and

$$\Theta_u\langle a, b\rangle \equiv u^{-1} T^\nabla \langle ua, ub\rangle \in T_oM.$$

Lemma 5.5 *Let $h \in H$, X^h be as in (3.3), then*

$$X^h b^\nabla \equiv \frac{d}{dt}\Big|_0 b^\nabla \circ e^{tX^h} = \int C^h \delta b^\nabla + h,$$

where

$$C_s^h \equiv \int_0^s \Omega_{u(s')}\langle h(s'), \delta b^\nabla(s')\rangle + \Theta_{u(s')}\langle h(s'), \cdot\rangle,$$

and $u(s) \equiv P_s^\nabla$.

Proof See Theorem 5.1 in [4]. Q.E.D.

Assuming sufficient regularity on b (i.e. on the "kernels" O and α) it is possible to compute $X^h b$ as:

$$\begin{aligned} X^h b &\equiv \frac{d}{dt}\Big|_0 b \circ e^{tX^h} \\ &= \int (X^h O) db^\nabla + \int O d(X^h b^\nabla) + \int (X^h \alpha) ds \\ &= \int (X^h O + OC^h) db^\nabla \quad \text{mod } H, \\ &= \int (X^h O + OC^h) O^{-1} db \quad \text{mod } H, \\ &= A(b)h + N(b)h, \end{aligned}$$

where

$$A(b)h \equiv \int (X^h O + OC^h) O^{-1} db \tag{5.3}$$

and $N(b) : H \to H$ is a bounded random linear operator on H. Working somewhat informally we have

$$\begin{aligned}\mathcal{E}^{\nabla}(f\circ b, f\circ b) &= \sum_{h\in S}\int_{W(M)} |X^h(f\circ b)|^2 d\nu \\ &= \sum_{h\in S}\int_{W(M)} (Df\circ b, X^h b)_H^2 d\nu \\ &= \sum_{h\in S}\int_{W(M)} (Df\circ b, (A(b)+N(b))h)_H^2 d\nu \\ &= \int_{W(M)} |(A(b)+N(b))^* Df\circ b|_H^2 d\nu.\end{aligned}$$

If this last expression is to be finite for all $f\in\mathcal{D}(Q)$, it seems likely that $A(b)^*$ and hence $A(b)$ must be a bounded linear operator on H. But this is only the case when $A(b)\equiv 0$, since otherwise $A(b)h$ is expressed as a stochastic integral relative to b. Since b is a Brownian motion on $(W(M),\tilde{\nu})$ where $\tilde{\nu}$ is a measure which is equivalent to ν, it follows that $A(b)h$ is not in H a.s. except when $A(b)h\equiv 0$.

For the reasons described above, we will try to choose (O,α) such that $A(b)\equiv 0$. By (5.3) $A(b)\equiv 0$ is equivalent to:

$$X^h O + OC^h \equiv 0 \ \forall h\in H. \tag{5.4}$$

In general a system of first order partial differential equations as in (5.4) will not have a solution unless the Frobenius integrability condition is satisfied. This condition is determined by formally requiring the equality of mixed partial derivatives of a supposed solution to (5.4). The next theorem describes the integrability conditions for (5.4). To avoid technical difficulties, I will state the theorem with $W(M)$ replaced by the Hilbert manifold $H(M)$ of finite energy paths in $W(M)$.

Theorem 5.6 *The differential equations in (5.4) are locally solvable on $H(M)$ iff $R^{\tilde{\nabla}}\equiv 0$, where $R^{\tilde{\nabla}}$ is the curvature tensor of the covariant derivative $\tilde{\nabla}$ on M defined by*

$$\tilde{\nabla}_X Y = \nabla_X Y - T^{\nabla}\langle X, Y\rangle. \tag{5.5}$$

Moreover, if $R^{\tilde{\nabla}}\equiv 0$ and $b\equiv b^{\tilde{\nabla}}$ then $\mathcal{E}^{\nabla}(f\circ b, f\circ b) = Q(f,f)$ for all $f\in\mathcal{D}(Q)$.

Remark 5.7 In the case that $M=G=SO(n)$ as in Example 2.1, then $\tilde{\nabla}^L=\nabla^R$, $\tilde{\nabla}^R=\nabla^L$, and $R^{\tilde{\nabla}^L}=R^{\nabla^R}\equiv 0$ and $R^{\tilde{\nabla}^R}=R^{\nabla^L}\equiv 0$. Thus Theorem 5.6 explains "why" Theorem 5.2 is valid.

Proof (I will sketch only the proof of the first statement in Theorem 5.6. A detailed proof will appear in future work.) Let $O(M)$ be the orthogonal frame bundle over M. (The fiber $O_m(M)$ of $O(M)$ over m is the set of isometries u from T_oM to T_mM.) Let $\tilde{\omega}$ be the connection 1-form on $O(M)$ corresponding to $\tilde{\nabla}$. More explicitly, for any path $u(s)\in O(M)$,

$$\tilde{\omega}\langle u'(s)\rangle \equiv u(s)^{-1}\tilde{\nabla}u(s)/ds.$$

Also let $C\langle\cdot\rangle$ denote the $L^2([0,1], so(n))$ – valued 1-form on $H^1(M)$ determined by $C\langle X^h\rangle = C^h$ for all $h\in H$, where $so(n)\equiv T_I SO(n)$ is the set of all real $n\times n$ skew symmetric matrices. (Recall that C^h was defined in Lemma 5.5.)

We now have the following facts:

1. $C=-(P^{\nabla})^*\tilde{\omega}$, where $(P^{\nabla})^*\tilde{\omega}$ denotes the differential form on $H^1(M)$ found by "pulling-back" $\tilde{\omega}$ by P^{∇}.
2. The integrability condition for (5.4) is equivalent to the statement that $-C$ has "zero curvature." That is

$$d(-C)+(-C)\wedge(-C)\equiv 0$$

Using these two facts, we learn that the integrability condition for (5.4) is equivalent to

$$0=(P^{\nabla})^*(d\tilde{\omega}+\tilde{\omega}\wedge\tilde{\omega})\equiv\tilde{\Omega}\langle P_*^{\nabla}\cdot, P_*^{\nabla}\cdot\rangle, \tag{5.6}$$

where $\tilde{\Omega}\equiv d\tilde{\omega}+\tilde{\omega}\wedge\tilde{\omega}$ is the equivariant form of the curvature tensor $R^{\tilde{\nabla}}$. Let $\pi: O(M)\to M$ denote the natural projection map and $e_s: W(M)\to M$ be the evaluation map: $e_s(\sigma)=\sigma(s)$. Since $\pi\circ P_s^{\nabla}=e_s$, it easily follows that $(P_s^{\nabla})_*$ maps $T_\sigma H^1(M)$ onto the "horizontal" vectors in $T_{P_s^{\nabla}(\sigma)}O(M)$. Since $\tilde{\Omega}$ annihilates "vertical" vectors it follows from (5.6) that $\tilde{\Omega}\equiv 0$ or equivalently $R^{\tilde{\nabla}}\equiv 0$. Q.E.D.

The integrability condition in Theorem 5.6 is very restrictive as the next theorem indicates. For the statement of the theorem it is necessary to recall the notion of a Killing vector field.

Definition 5.8 A *Killing vector field* on M is a vector field X on M such that the flow e^{tX} preserves the Riemannian metric $\langle\cdot,\cdot\rangle$. That is $\langle e_*^{tX}v, e_*^{tX}v\rangle=\langle v,v\rangle$ for all $v\in TM$.

Theorem 5.9 *Assume that M is a simply connected Riemannian manifold with Riemannian metric $\langle\cdot,\cdot\rangle$. Then there exists a covariant derivative ∇ on TM such* *that $R^{\tilde{\nabla}} \equiv 0$ ($\tilde{\nabla}$ as in (5.5)) iff there exists a global orthonormal frame of Killing vector fields $\{X_i\}_{i=1}^d$ on M. In particular, M is a homogeneous space with a trivial tangent bundle.*

Example 5.10 Suppose that $G = SO(n)$ and $\{A_i\}_{i=1}^d$ ($d = n(n-1)/2$) is an orthonormal basis for $so(n) \equiv T_I SO(n)$. Set $X_i^L(g) \equiv gA_i$ and $X_i^R(g) \equiv A_i g$ for all $g \in G$. Then $\{X_i^L\}_{i=1}^d$ and $\{X_i^R\}_{i=1}^d$ are two orthonormal frames of Killing vector fields on G.

This example and its minor generalization to Lie groups of compact type are the only examples that I know of manifolds which admit an orthonormal basis of Killing vector-fields. It is well known that the generic Riemannian manifold does not admit any Killing vector fields, see Bochner [3]. In conclusion, it seems that the Dirichlet forms constructed in (3.4) are typically non-equivalent to the usual Dirichlet form Q on W (at least if the map $b : W(M) \to W$ is required to be adapted.)

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