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Integration by Parts and Quasi-Invariance for Heat Kernel Measures on Loop Groups

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Integration by parts formulas are established both for Wiener measure on the path space of a loop group and for the heat kernel measures on the loop group. The Wiener measure is defined to be the law of a certain loop group valued "Brownian motion" and the heat kernel measures are time t, $t>0$, distributions of this Brownian motion. A corollary of either of these integrations by parts formulas is the closability of the pre-Dirichlet form considered by B. K. Driver and T. Lohrenz [1996, *J. Functional Anal.* **140**, 381–448]. We also show that the heat kernel measures are quasi-invariant under right under right and left translations by finite energy loops. © 1997 Academic Press

Contents

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1. INTRODUCTION

Let G be a connected compact[1] Lie group equipped with an Ad_G-invariant inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra $(\mathfrak{g} \equiv T_e G)$ of G. Let $\mathscr{L}(G)$ denote the space of continuous loops in G based at the identity. Following Malliavin [17], a $\mathscr{L}(G)$-valued processes $\{\Sigma_t\}_{t \geqslant 0}$ is constructed; see Theorem 3.8. In Theorem 3.10 below this processes is shown to satisfy the martingale characterization of a Brownian motion on $\mathscr{L}(G)$. Let $v \equiv \mathrm{Law}(\Sigma_{(\cdot)})$ and $v_T \equiv \mathrm{Law}(\Sigma_T)$ so that v (Wiener measure) and v_T (heat kernel measure) are probability measures on the path space of $\mathscr{L}(G)$ and $\mathscr{L}(G)$, respectively. Two types of integration by parts formulas are established.

The first integration by parts formula is for the measure v relative to a certain class of vector fields on the path space. This version is an "infinite" dimensional version of the integration by parts theorem in Driver [4], see Theorem 9.1 on p. 363.

The second is for the left-invariant first order differential operators on $\mathscr{L}(G)$. This version is a infinite dimensional analogue of the fact that heat kernel on a finite dimensional Lie group has a logarithmic derivative. Of course, the finite dimensional version follows from the fact that the heat kernel measure is absolutely continuous relative to the Riemannian volume measure and the Radon–Nikodym density is smooth and never zero.

In Driver and Lohernz [6], a Logarithmic Sobolev inequality for cylinder functions was proved on a loop group with the underlying reference measure being the heat kernel measure v_T. The Logarithmic Sobolev inequality as stated in [6] is really a collection of Logarithmic Sobolev inequalities for certain finite dimensional approximations to the Loop group with the constants being independent of the approximation. A corollary of either of the integrations by parts formulas in this paper is that the pre-Dirichlet form considered in [6] is closable. This elevates the Logarithmic Sobolev inequality in [6] to a truly infinite dimensional inequality.

We will also show that the heat kernel measure v_T is quasi-invariant under right and left translations by "finite energy" loops in $\mathscr{L}(G)$. This will be done using an argument due to Cruzeiro [2] (see also Dennis Bell [1] and Gunnar Peters [20, 21]) for proving quasi-invariance of flow from integration integration by parts formulas.

[1] To avoid certain technical complications, G is assumed to be compact in the body of this paper. However, it would be possible to extend the results in this paper to the case where G is a Lie group of compact type, i.e., $G = K \times \mathbb{R}^d$, where K is a compact Lie group and $d \in \{0, 1, 2, ...\}$.

1.1. *Statement of Results*

Let G be a compact Lie group, $\mathfrak{g} \equiv T_e G$ be the Lie algebra of G, and $\langle \cdot, \cdot \rangle$ be an Ad_G invariant inner product on $\mathfrak{g}$. Let $\mathscr{L} = \mathscr{L}(G)$ denote the based loop group on G consisting of continuous paths $g: [0, 1] \to G$ such that $g(0) = g(1) = e$, where $e \in G$ is the identity element. Similarly, $\mathscr{L}(\mathfrak{g})$ will denote the continuous paths $h: [0, 1] \to \mathfrak{g}$ such that $h(0) = h(1) = 0$.

Given $h \in \mathscr{L}(\mathfrak{g})$, define $(h, h) = \infty$ if h is not absolutely continuous and set $(h, h) = \int_0^1 |h'(s)|^2 \, ds$ otherwise. Let

$$H_0(\mathfrak{g}) \equiv \{h: [0, 1] \to \mathfrak{g} \mid h(0) = h(1) = 0 \text{ and } (h, h) < \infty\}. \tag{1.1}$$

Hence $H_0(\mathfrak{g}) \subset \mathscr{L}(\mathfrak{g})$ is Hilbert space with inner product $(h, k) = \int_0^1 \langle h'(s), k'(s) \rangle \, ds$. The Hilbert space $H_0(\mathfrak{g})$ is to be thought of as the Lie algebra of $\mathscr{L}(G)$. Using left translation, we may extend the inner product $(\cdot, \cdot)$ to a "Riemannian metric" on the Cameron-Martin tangent space $(T\mathscr{L})$ to $\mathscr{L}$. Explicitly,

$$T\mathscr{L} \equiv \{X: [0, 1] \to TG \mid \theta\langle X \rangle \in H_0(\mathfrak{g})\}, \tag{1.2}$$

where $(\theta\langle X \rangle)(s) \equiv \theta\langle X(s) \rangle$ and θ is the Maurer Cartan form on G, i.e., $\theta\langle \xi \rangle = L_{g^{-1}*} \xi \in \mathfrak{g}$ for all $\xi \in T_g G$ and $g \in G$. Let $\pi: TG \to G$ denote the projection of a tangent verctor in TG to its base point. Given $g \in \mathscr{L}$, the tangent space to $\mathscr{L}$ at g is

$$T_g \mathscr{L} \equiv \{X \in T\mathscr{L} : \pi \circ X = g\} \subset T\mathscr{L}.$$

The length (X, X) of a tangent vector $X \in T\mathscr{L}$ is now defined by

$$(X, X) \equiv (\theta\langle X \rangle, \theta\langle X \rangle)_{H_0(\mathfrak{g})}.$$

In this way, $\mathscr{L}$ is to be thought of as an infinite dimensional "Riemannian" manifold.

The Levi-Civita covariant derivative (D) acting on $H_0(\mathfrak{g})$, which should be identified with left-invariant vector fields on $\mathscr{L}$, is determined by

$$(D_k h)(s) \equiv \int_0^s [k(\sigma), h'(\sigma)] \, d\sigma - s \int_0^1 [k(\sigma), h'(\sigma)] \, d\sigma, \tag{1.3}$$

where $h, k \in H_0(\mathfrak{g})$. See Proposition 1.6 in Freed [10] and Definition 3.6 and Theorem 3.12 in Driver and Lohernz [6]. As for finite dimensional Lie groups,[2] Eq. (1.3) uniquely determines the Levi-Civita covariant derivative ∇ acting on paths in $T\mathscr{L}$. Namely, if $t \to X(t)$ is path in $T\mathscr{L}$ such that

[2] For the case of finite dimensional Lie groups see Section 6 in [5].

$h(t) \equiv \theta\langle X(t)\rangle \in H_0(\mathfrak{g})$ and $g(t) \equiv \pi \circ X(t) \in \mathscr{L}$ are sufficiently smooth paths, then Levi-Civita covariant derivative of $X(\cdot)$ is

$$\nabla X(t)/dt = L_{g(t)*}\{\dot{h}(t) + D_{\dot{\beta}(t)} h(t)\},$$

where $\dot{\beta}(t) \equiv \theta\langle \dot{g}(t)\rangle \in H_0(\mathfrak{g})$. In particular, parallel translation ($//$) along a sufficiently smooth path $t \to g(t) \in \mathscr{L}(G)$ is defined by $//_t = L_{g(t)*} U(t)$, where U solves the ordinary differential equation,

$$\frac{dU(t)}{dt} + D_{\dot{\beta}(t)} U(t) = 0 \qquad \text{with} \quad U(0) = I_{H_0(\mathfrak{g})}. \tag{1.4}$$

Let $\{\beta(t)\}_{t \geq 0}$ be an $\mathscr{L}(\mathfrak{g})$-valued Brownian motion with covariance determined by Hilbert norm $(\cdot, \cdot)$. A more precise description of β is that $\beta = \{\beta(t, s)\}_{t \geq 0,\, s \in [0, 1]}$ is a jointly continuous two parameter $\mathfrak{g}$-valued Gaussian process with mean zero and covariance given by

$$E[\langle A, \beta(t, s)\rangle\langle B, \beta(\tau, \sigma)\rangle] = \langle A, B\rangle(t \wedge \tau)(s \wedge \sigma - s\sigma),$$

where $A, B \in \mathfrak{g}$, $t, \tau \in [0, \infty)$, $s, \sigma \in [0, 1]$, and $s \wedge \sigma \equiv \min(s, \sigma)$. (See Section 3.1 for a more detailed discussion.) Following Malliavin [17], we have the following theorem which is proved in Section 3 below, see Theorem 3.8.

Theorem 1.1 (Brownian Motion on $\mathscr{L}$). *Given $g_0 \in \mathscr{L}(G)$, there is a jointly continuous solution $\Sigma(t, s)$ to the stochastic differential equation*

$$\Sigma(\delta t, s) = L_{\Sigma(t, s)*} \beta(\delta t, s) \qquad \textit{with} \quad \Sigma(0, s) = g_0(s) \qquad \forall s \in [0, 1], \tag{1.5}$$

where for each fixed $s \in [0, 1]$, $\Sigma(\delta t, s)$ and $\beta(\delta t, s)$ denote the Stratonovich differentials of the processes $t \to \Sigma(t, s)$ and $t \to \beta(t, s)$ respectively. (In the sequel, for concreteness we will assume that Σ is the process defined in Eq. (1.5) above with $g_0(s) \equiv e$ for all $0 \leq s \leq 1$.)

Notation 1.2. The *Wiener space based on* $\mathscr{L} = \mathscr{L}(G)$ is the set of paths

$$W(\mathscr{L}) \equiv \{\sigma \in C([0, \infty), \mathscr{L}) : \sigma(0) = e \in \mathscr{L}\}. \tag{1.6}$$

Similarly, let $H(H_0(\mathfrak{g}))$ be the set of continuous functions $h: [0, \infty) \to H_0(\mathfrak{g})$ such that $h(0) = 0$ and there is a function $\dot{h} \in L^2([0, \infty), dt; H_0(\mathfrak{g}))$ such that $h(t) = \int_0^t \dot{h}(\tau)\, d\tau$ for all $t \in [0, \infty)$. (The integral is taken to be the Bochner integral. As in the scalar valued case, one may show that such a function h is absolutely continuous, the derivative of h exists for almost every $t \in [0, \infty)$, and $dh(t)/dt = \dot{h}(t)$ a.e.) Then $H(H_0(\mathfrak{g}))$ becomes a Hilbert space with inner product defined by

$$(h, k)_{H(H_0(\mathfrak{g}))} \equiv \int_0^\infty (\dot{h}(t), \dot{k}(t))_{H_0(\mathfrak{g})}\, dt$$

for all $h, k \in H(H_0(\mathfrak{g}))$.

Definition 1.3 (Cylinder Functions). A function $f\colon \mathcal{L} \to \mathbb{R}$ is said to be a *smooth cylinder function on* $\mathcal{L}$ if f has the form

$$f(g) = F(g(s_1), ..., g(s_n)) \tag{1.7}$$

for some partition $\mathcal{P} = \{0 < s_1 < s_2 < \cdots < s_n < 1\}$ of $[0, 1]$ and some $F \in C^\infty(G^n)$. The collection of smooth cylinder functions on $\mathcal{L}$ will be denoted by $\mathcal{F}C^\infty(\mathcal{L})$. A function $f\colon W(\mathcal{L}) \to \mathbb{R}$ is said to be a *smooth cylinder function on* $W(\mathcal{L})$ if f can be written in the form

$$f(\sigma) = F(\sigma(t_1, s_1), ..., \sigma(t_n, s_n)) \qquad \forall \sigma \in W(\mathcal{L}), \tag{1.8}$$

where $F \in C^\infty(G^n)$ and $\{(t_i, s_i)\}_{i=1}^n \subset [0, \infty) \times (0, 1)$. The collection of smooth cylinder functions on $W(\mathcal{L})$ will be denoted by $\mathcal{F}C^\infty(W(\mathcal{L}))$.

To simplify notation in the sequel we will let

$$g_{\mathcal{P}} \equiv (g(s_1), ..., g(s_n)) \tag{1.9}$$

when $g \in \mathcal{L}$ and $\mathcal{P} = \{0 < s_1 < s_2 < \cdots < s_n < 1\}$. With this notation Eq. (1.7) may be written as $f(g) = F(g_{\mathcal{P}})$.

Theorem 3.10 below shows that the process $\Sigma_t \equiv \Sigma(t) \equiv \Sigma(t, \cdot) \in \mathcal{L}(G)$ deserves to be called Brownian motion on $\mathcal{L}(G)$ starting at g_0. Let ν denote the law of $\{\Sigma(t, s)\}_{t \geqslant 0,\, s \in [0, 1]}$ and ν_T denote the law of $\Sigma_T \equiv \Sigma(T, \cdot)$. Also let $//_t$ be "stochastic parallel translation" along the Brownian motion $\Sigma(t)$. In analogy to the smooth case as above, $//_t \equiv L_{\Sigma(t)*} U(t)$, where $U(t)$ is process taking values in the unitary group of $H_0(\mathfrak{g})$ which "solves" the Stratonovich stochastic differential equation,

$$\delta U(t) + D_{\beta(\delta t)} U(t) = 0, \qquad \text{with} \quad U(0) = I_{H_0(\mathfrak{g})}. \tag{1.10}$$

See Theorem 4.1, Definition 4.2, Lemma 4.3 and the discussion at the beginning of Section 4 for more details. The following integration by parts theorem for $W(\mathcal{L})$ is completely analogous to the well known integration by parts theorem (see for example Theorem 9.1 in [4]) for the Wiener space $W(M)$ of compact Riemannian manifold M.

THEOREM 1.4 (Integration by Parts on $W(\mathscr{L})$). *For each* $h \in H(H_0(\mathfrak{g}))$ *let* X^h *denote the vector-field on* $W(\mathscr{L}(G))$ *defined by*

$$X_t^h(\Sigma) = //_t h(t) = L_{\Sigma(t)*} U(t)\, h(t).$$

Then for all smooth cylinder functions f *on* $W(\mathscr{L})$

$$E[(X^h f)] = E[f(\Sigma(T))\, z_T(h)], \tag{1.11}$$

where

$$(X^h f)(\Sigma) \equiv \frac{d}{du}\bigg|_0 f(\Sigma e^{uX^h(\Sigma)}), \tag{1.12}$$

and $z_T(h)$ *is a random variable described in Eq.* (4.17) *below.*

This theorem is proved in Section 4 using the method which has been described in Hsu [12] and Sections 2 and 3 of Driver [5] when the loop group is replaced by a finite dimensional Riemannian manifold, see Theorems 4.10 and 4.12 below. The next theorem describes an integration by parts formula for the left invariant vector fields on $\mathscr{L}$.

THEOREM 1.5 (Integration by Parts on $\mathscr{L}$). *Let* $t > 0$, $h \in H_0(\mathfrak{g})$, f *be a cylinder function on* $\mathscr{L}$, *and* $\tilde{h} f(g) \equiv (d/dt)|_0 f(ge^{th})$. (*So* $\tilde{h}$ *is a first order left invariant differential operator on* $\mathscr{L}$.) *Define* $H(\tau)$, $\tau \in [0, t]$, *to be the solution to the Stratonovich stochastic differential equation*:

$$dH(\tau) + D_{\delta\beta(\tau)} H(\tau) = 0 \qquad \text{with final data} \quad H(t) = h. \tag{1.13}$$

(*The precise meaning of this equation is explained in Theorem* 6.1 *below.*) *Then*

$$E[(\tilde{h} f)(\Sigma_t)] = \frac{1}{t} E\left[f(\Sigma_t) \int_0^t \left(\left\{ I - \frac{1}{2} \tau \operatorname{Ric} \right\} H(\tau)\, \overleftarrow{d\beta}(\tau) \right) \right], \tag{1.14}$$

where $\overleftarrow{d\beta}$ *in Eq.* (1.14) *denotes the backwards stochastic differential and* Ric *is the Ricci tensor on* $\mathscr{L}$. *See Section* 8.3 *of the Appendix for a short review of the backwards Itô integral and Definition* 2.4 *below for the meaning of the Ricci tensor* Ric.

Theorem 1.5 is a special case of Theorem 6.2 below. Theorems 1.5 and 6.2 turn out to be more delicate than Theorem 1.4. The proof is based on Corollary 6.4 in Driver [5], which is a finite dimensional analogue to Theorem 1.5. The basic idea of the proof is to apply Corollary 6.4 in [5] to certain finite dimensional approximations to the loop group and then to pass to the limit of finer and finer approximations. The necessary geometry and estimates for the finite dimensional approximations, which are needed to carry out this limiting procedure, are developed in Section 5. See in particular Theorems 5.8 and 5.10.

An application of either of the above integration by parts formulas is the closability (see Theorem 4.14 below) of the symmetric pre-Dirichlet form on $L^2(\mathcal{L}(G), \nu_T)$ defined as: $\mathcal{D}(\mathcal{E}^0) = \mathcal{F}C^\infty(\mathcal{L})$ and for $f \in \mathcal{D}(\mathcal{E}^0)$,

$$\mathcal{E}^0(f, f) \equiv \int_{\mathcal{L}} \|\vec{\nabla} f(g)\|^2_{H_0(\mathfrak{g})} \, \nu_T(dg).$$

Here $\vec{\nabla} f(g)$ denotes the gradient of f at $g \in \mathcal{L}$, i.e., $\vec{\nabla} f(g)$ is the unique element in $H_0(\mathfrak{g})$ such that

$$(\vec{\nabla} f(g), h) = (\tilde{h} f)(g) \qquad \forall h \in H_0(\mathfrak{g}). \tag{1.15}$$

A second application of Theorem 1.5 is the quasi-invariance of the heat kernel measure ν_T under left and right translations by "finite energy" loops in $\mathcal{L}(G)$, see Corollary 7.7 and 7.10 in Section 7 below. The quasi-invariance under right translations by finite energy loops will be proved using the second integration by parts formula coupled with an argument due to Cruzeiro [2] (see also Dennis Bell [1]) for proving quasi-invariance of flows from integration by parts formulas. The quasi-invariance under left translations by finite energy loops then follows easily from the fact that ν_T is invariant under the transformation $g \in \mathcal{L}(G) \to g^{-1} \in \mathcal{L}(G)$, see Proposition 7.9 below.

2. NOTATION AND PREREQUISITES

This section gathers some needed additional notation and results from Driver and Lohrenz [6] and in Driver [5]. Let $HS(H_0(\mathfrak{g})) \cong H_0(\mathfrak{g})^* \otimes H_0(\mathfrak{g})$ be the Hilbert Schmidt operators on $H_0(\mathfrak{g})$, $S_0 \subset H_0(\mathfrak{g})$ be an orthonormal basis for $H_0(\mathfrak{g})$ and $\mathfrak{g}_0 \subset \mathfrak{g}$ be an orthonormal basis of $\mathfrak{g}$. For $A \in \mathfrak{g}$, let $\tilde{A}$ be the unique left invariant vector field on G such that $\tilde{A}(e) = A$. The following theorem may be found in Lemma 3.9 and Theorem 3.12 of [6].

THEOREM 2.1. *For $k \in H_0(\mathfrak{g})$, let $D_k \colon H_0(\mathfrak{g}) \to H_0(\mathfrak{g})$ denote the operator defined in Eq.* (1.3). *We will also think of D as an operator from $H_0(\mathfrak{g}) \to HS(H_0(\mathfrak{g}))$ via $(Dh)\,k \equiv D_k h$. Then D is a bounded operator such that*

$$\|D\|^2_{op} \equiv \sup_{\|h\|_{H_0(\mathfrak{g})} = 1} \sum_{k \in S_0} \|D_k h\|^2 < \infty$$

and D_k is skew adjoint operator on $H_0(\mathfrak{g})$ for all $k \in H_0(\mathfrak{g})$.

If T is a Hilbert space, we will say that $f\colon \mathcal{L} \to T$ is a smooth cylinder function if f has the form

$$f(g) = \sum_{i=1}^{n} f_i(g)\, x_i \qquad \forall g \in \mathcal{L},$$

where $f_i \in \mathcal{F}C^\infty(\mathcal{L})$ and $x_i \in T$. The set of smooth cylinder functions on $\mathcal{L}$ with values in T will be denoted by $\mathcal{F}C^\infty(\mathcal{L}, T)$. The left invariant vector fields $\tilde{h}$ for $h \in H_0(\mathfrak{g})$ extend naturally to operators on $\mathcal{F}C^\infty(\mathcal{L}, T)$, namely

$$\tilde{h}f(g) \equiv \frac{d}{dt}\bigg|_0 f(ge^{th}).$$

Definition 2.2 (Covariant Derivative). Let $h \in H_0(\mathfrak{g})$. Define ∇_h via:

1. if $f \in \mathcal{F}C^\infty(\mathcal{L})$, set $\nabla_h f \equiv \tilde{h}f$.
2. If $f \in \mathcal{F}C^\infty(\mathcal{L}, H_0(\mathfrak{g}))$, set $\nabla_h f \equiv \tilde{h}f + D_h f$, where $(D_h f)(g) \equiv D_h(f(g))$.
3. If $f \in \mathcal{F}C^\infty(\mathcal{L}, H_0(\mathfrak{g})^*)$, set $\nabla_h f \equiv \tilde{h} - D_h^{tr} f$, where $(D_h^{tr} f)(g) \equiv D_h^{tr}(f(g))$ and $D_h^{tr}\colon H_0(\mathfrak{g})^* \to H_0(\mathfrak{g})^*$ is the transpose of the operator D_h; i.e., $D_h^{tr} l \equiv l \circ D_h$ for $l \in H_0(\mathfrak{g})^*$.

Definition 2.3 (Laplacian). For $f \in \mathcal{F}C^\infty(\mathcal{L})$ or $f \in \mathcal{F}C^\infty(\mathcal{L}, H_0(\mathfrak{g}))$ or $f \in \mathcal{F}C^\infty(\mathcal{L}, H_0(\mathfrak{g})^*)$, the *Laplacian* of f is defined by

$$\Delta f \equiv \sum_{h \in S_0} \nabla_h^2 f \equiv \sum_{h \in S_0} \nabla_h(\nabla_h f). \tag{2.1}$$

The existence of the above sum is guaranteed by Proposition 4.19 of [6]. We now introduce the Ricci tensor on $\mathcal{L}$, see Freed [10] and Driver and Lohrenz [6] for more details and motivation. This tensor naturally appears in all of the integration by parts formulas that we consider.

Definition 2.4 (Ricci Tensor). The Ricci tensor is the symmetric quadratic form on $H_0(\mathfrak{g})$ defined by

$$\mathrm{Ric}\langle h, k\rangle = -\int_0^1 \int_0^1 G_0(\sigma, s)\, K\langle h'(s), k'(\sigma)\rangle\, d\sigma\, ds \qquad \forall h, k \in H_0(\mathfrak{g}), \tag{2.2}$$

where $G_0(\sigma, s) \equiv \sigma \wedge s - s\sigma$ and

$$K\langle B, C\rangle \equiv \sum_{A \in \mathfrak{g}_0} \langle ad_A B, ad_A C\rangle = -\mathrm{tr}(ad_B ad_C),$$

for all $B, C \in \mathfrak{g}$. That is K is the negative of the Killing form on $\mathfrak{g}$. We will also view Ric as a bounded symmetric linear operator on $H_0(\mathfrak{g})$, explicitly $(\mathrm{Ric}\, h, k) = \mathrm{Ric}\langle h, k\rangle$.

The following theorem summarizes the properties of the gradient (defined in Eq. (1.15)), the Laplacian and the Ricci tensor that we will need in the sequel.

THEOREM 2.5. *Let $f \in \mathscr{F}C^{\infty}(\mathscr{L})$, be given as in Eq. (1.7), then*

$$\Delta f(g) = \sum_{A \in \mathfrak{g}_0} \sum_{i, j=1}^{n} G_0(s_i, s_j)(A^{(i)} A^{(j)} F)(g(s_1), ..., g(s_n)), \tag{2.3}$$

and

$$\vec{\nabla} f(g) = \sum_{A \in \mathfrak{g}_0} \sum_{i=1}^{n} (A^{(i)} F)(g_{\mathscr{P}})\, G_0(s_i, \cdot)\, A, \tag{2.4}$$

where for $A \in \mathfrak{g}$, $A^{(i)}$ is the left invariant vector-field on G^n defined by

$$(A^{(i)} F)(g_1, ..., g_n) \equiv \frac{d}{d\varepsilon}\Big|_0 F(g_1, ..., g_{i-1}, g_i e^{\varepsilon A}, g_{i+1}, g_n). \tag{2.5}$$

(As above, $\mathfrak{g}_0 \subset \mathfrak{g}$ is an orthonormal basis of $\mathfrak{g}$.)

The Bochner Wietzenbock *formula in this context is*

$$([\Delta, \vec{\nabla}]\, f) \equiv \Delta \vec{\nabla} f - \vec{\nabla} \Delta f = \mathrm{Ric}\, \vec{\nabla} f. \tag{2.6}$$

If $H_0(\mathfrak{g})$ is viewed as the subspace of constant functions in $\mathscr{F}C^{\infty}(\mathscr{L}, H_0(\mathfrak{g}))$, then

$$\Delta^{(1)} \equiv \Delta|_{H_0(\mathfrak{g})} = \sum_{k \in S_0} D_k^2.$$

This sum is strongly convergent and $\Delta^{(1)}$ is a bounded self-adjoint operator on $H_0(\mathfrak{g})$.

Proof. See [6] Proposition 4.19 for Equation (2.3), Theorem 4.26 for Eq. (2.6), and Lemma 4.20 for the assertions concerning $\Delta^{(1)}$. Equation (2.4) is easily checked using the definition of $\vec{\nabla} f$ in Eq. (1.15) and the reproducing kernel property of G_0, see Eq. (3.11) in [6] or the discussion preceding Eq. (3.3) below. Q.E.D.

3. BROWNIAN MOTION ON LOOP GROUPS

Let $\mathcal{L}(\mathfrak{g}) \equiv \{x \in C([0,1] \to \mathfrak{g} \mid x(0) = x(1) = 0\}$ be the continuous based loops in $\mathfrak{g}$. It is well known that $(H_0(\mathfrak{g}), \mathcal{L}(\mathfrak{g}))$ is an abstract Wiener space as introduced by Gross in [11]. As usual in the abstract Wiener space setting, we have $\mathcal{L}(\mathfrak{g})^* \subset H_0(\mathfrak{g})^* \cong H_0(\mathfrak{g}) \subset \mathcal{L}(\mathfrak{g})$. Let us recall the explicit description of $\mathcal{L}(\mathfrak{g})^*$ in $H_0(\mathfrak{g})^*$.

To this end we will say, for $h \in H_0(\mathfrak{g})$, that h' is of *bounded variation* if there is a right continuous function (λ) of bounded variation such that $h'(s) = \lambda(s)$ a.e.. Let

$$H_0^{BV} \equiv \{h \in H_0 \mid h' \text{ is of bounded variation}\}.$$

Now suppose that $k \in H_0(\mathfrak{g})$, then by an integration by parts (see for example Theorem 3.30 of [9])

$$(h, k) = \int_0^1 \langle h'(s), k'(s)\rangle \, ds = \int_0^1 \langle \lambda(s), dk(s)\rangle = -\int_0^1 \langle k(s), d\lambda(s)\rangle. \quad (3.1)$$

In the future we will abuse notation and write $\int_0^1 \langle k(s), dh'(s)\rangle$ for $\int_0^1 \langle k(s), d\lambda(s)\rangle$.

Lemma 3.1. *For each $h \in H_0^{BV}$ and $x \in \mathcal{L}(\mathfrak{g})$ let $\alpha_h(x) \equiv -\int_0^1 \langle x(s), dh'(s)\rangle$. Then the map $h \in H_0^{BV} \to \alpha_h \in \mathcal{L}(\mathfrak{g})^*$ is an isomorphism. Moreover $\alpha_h(k) = (h, k)$ for all $k \in H_0(\mathfrak{g})$.*

Proof. The last assertion of the Lemma clearly follows from (3.1). Now suppose that $\alpha_h \equiv 0$ then $0 = \alpha_h(k) = (h, k)$ for all $k \in H_0(\mathfrak{g})$ which implies that $h = 0$ in $H_0(\mathfrak{g})$. Therefore $h \to \alpha_h$ is injective.

Since $\mathcal{L}(\mathfrak{g})^* \subset H_0(\mathfrak{g})^*$, for $\alpha \in \mathcal{L}(\mathfrak{g})^*$ there exists $h \in H_0(\mathfrak{g})$ such that $\alpha(k) = (h, k)$ for all $k \in H_0(\mathfrak{g})$. Since $H_0(\mathfrak{g})$ is dense in $\mathcal{L}(\mathfrak{g})$, if we can show that $h \in H_0^{BV}$, it will follow that $\alpha = \alpha_h$. Hence the map $h \to \alpha_h$ is surjective.

Noting that $\mathcal{L}(\mathfrak{g})$ is a closed subspace of $C([0,1], \mathfrak{g})$, the Hahn-Banach theorem asserts that α has an extension $(\tilde{\alpha})$ to a bounded linear functional on $C([0,1], \mathfrak{g})$. By the Riesz theorem (e.g., Theorem 7.17 of [9]) there is a $\mathfrak{g}$-valued measure μ such that

$$\tilde{\alpha}(x) = \int_0^1 \langle x(s), \mu(ds)\rangle \qquad \forall x \in C([0,1], \mathfrak{g}).$$

Define $\lambda(s) \equiv \mu([0, s]) \in \mathfrak{g}$ for $s \in [0,1]$. Then λ is of bounded variation and we have

$$\alpha(x) = \int_0^1 \langle x(s), d\lambda(s)\rangle \qquad \forall x \in \mathcal{L}(\mathfrak{g}).$$

Restricting this last identity to $k \in H_0(\mathfrak{g})$ and then doing an integration by parts shows that

$$\int_0^1 \langle h'(s), k'(s)\rangle\, ds = (h, k) = \alpha(k) = \int_0^1 \langle k(s), d\lambda(s)\rangle$$
$$= -\int_0^1 \langle \lambda(s), k'(s)\rangle\, ds.$$

Since $\{k' \in L^2([0,1], \mathfrak{g}) \mid k \in H_0(\mathfrak{g})\}$ is the orthogonal compliment of the constant functions in $L^2([0,1], \mathfrak{g})$ the above equation implies that $h'(s) = \lambda(s) - \int_0^1 \lambda(s)\, ds$ a.e. This proves that $h \in H_0^{BV}$. Q.E.D.

Notation 3.2. In the sequel, we will write (h, x) instead of $\alpha_h(x)$ when $h \in H_0^{BV}$ and $x \in \mathscr{L}(\mathfrak{g})^*$.

3.1. $\mathscr{L}(\mathfrak{g})$-*Valued Brownian Motion*

Let $\mathscr{G}$ denote the smallest σ-field on $\mathscr{L}$ such that all of the smooth cylinder functions in $\mathscr{F}C^\infty(\mathscr{L})$ are measurable. For the sequel, fix a filtered probability space $(\mathscr{W}, \{\mathscr{F}_t\}_{t \geqslant 0}, \mathscr{F}, P)$ and a $\mathscr{L}(\mathfrak{g})$-valued process $\{\beta(t)\}_{t>0}$ on $\mathscr{W}$ with the following properties:

1. $\mathscr{F}_t \subset \mathscr{F}_{t'} \subset \mathscr{F}$ for all $0 \leqslant t \leqslant t'$.
2. $\mathscr{F}_t$ is right continuous, i.e., $\mathscr{F}_t = \mathscr{F}_{t+} \equiv \bigcap_{\varepsilon>0} \mathscr{F}_{t+\varepsilon}$,
3. $\mathscr{F}_0$ contains all of the null sets of $\mathscr{F}$,
4. for all $\omega \in \mathscr{W}$, the map $t \in [0, \infty) \to \beta(t)(\omega) \in \mathscr{L}(\mathfrak{g})$ is continuous.
5. $\beta(t)$ is $\mathscr{F}_t/\mathscr{G}$-measurable for all $t \geqslant 0$, and
6. $\{\beta(t)\}_{t \geqslant 0}$ is a mean-zero Gaussian process with convariance,

$$E[(h, \beta(t))(k, \beta(\tau))] = t \wedge \tau (h, k), \tag{3.2}$$

where $h, k \in H_0^{BV}$ and $t, \tau \in [0, \infty)$.

We say that such a process $\{\beta(t)\}_{t \geqslant 0}$ is a $\mathscr{L}(\mathfrak{g})$-valued *Brownian motion.*

Remark 3.3. The existence of an $\mathscr{L}(\mathfrak{g})$-valued Brownian motion $\beta(t)$ is well known. In fact, it is known more generally that Brownian motions exist on arbitrary abstract Wiener spaces. One possible construction is to first use Kolomogorov's existence theorem to construct a Brownian motion $\tilde{\beta}(t)$ satisfying all of the properties above except for the continuity. Then by Fernique's theorem (e.g., Theorem 3.1 of Kuo [15]) and scaling it can be seen that Kolomogorov's continuity criteria may be applied to yield a version $\beta(t)$ of $\tilde{\beta}(t)$ which is α holder continuous for all $\alpha \in (0, 1/2)$.

Suppose that $h(s) \equiv G_0(s, u) A$ where $A \in \mathfrak{g}$ and as above $G_0(s, u) = s \wedge u - su$. Then for $x \in \mathscr{L}(\mathfrak{g})$

$$(h, x) = -\int_0^1 \langle x(s), dh'(s) \rangle = -\int_0^1 \langle x(s), A \rangle \, d_s\{1_{s \leqslant u} - u\}$$

$$= -\int_0^1 \langle x(s), A \rangle (-\delta_u(ds)) = \langle x(u), A \rangle,$$

where δ_u is the Dirac measure concentrated at u. Let $k(s) \equiv G_0(s, v) B$. Notice that $(h, k) = \langle A, B \rangle \, G_0(u, v)$.

Write $\beta(t, s)$ for the $\mathfrak{g}$-valued random variable determined by $\beta(t, s)(\omega) \equiv \beta(t)(\omega)(s)$. Using the previous paragraph and (3.2), for all $A, B \in \mathfrak{g}$ and $t, \tau \in [0, \infty)$ with $t \geqslant \tau$,

$$\begin{aligned} &E[\langle A, \beta(t, u) \rangle \langle B, \beta(\tau, v) \rangle] \\ &\quad = E[\{\langle A, \beta(\tau, u) \rangle + \langle A, \beta(t, v) - \beta(\tau, u) \rangle\} \langle B, \beta(\tau, v) \rangle] \\ &\quad = E[\langle A, \beta(\tau, u) \rangle \langle B, \beta(\tau, v) \rangle] \\ &\quad = \langle A, B \rangle \, \tau G_0(u, v) \\ &\quad = \langle A, B \rangle (t \wedge \tau) \, G_0(u, v). \end{aligned} \tag{3.3}$$

For each $h \in H_0(\mathfrak{g})$ and $t \geqslant 0$, let

$$(h, \beta(t)) \equiv L^2\text{-} \lim_{n \to \infty} (h_n, \beta(t)),$$

where $\{h_n\} \subset H_0^{BV}$ and $h_n \to h$ in $H_0(\mathfrak{g})$ as $n \to \infty$. Then it is easily checked that $t \to (h, \beta(t))$ is a (not necessarily continuous) Brownian motion with variance (h, h). Let $\beta^h(t)$ denote a continuous version of $(h, \beta(t))$. Such a version exists by Kolomogorov's continuity criteria. Then β^h is a Brownian motion with variance (h, h) on the filtered probability space $(\mathscr{W}, \{\mathscr{F}_t\}, \mathscr{F}, P)$. The next Lemma records the mutual quadratic variation $\prec \beta^h, \beta^k \succ$ for $h, k \in H_0(\mathfrak{g})$.

LEMMA 3.4. *For each $h, k \in H_0(\mathfrak{g})$,*

$$\prec \beta^h, \beta^k \succ_t = (h, k) \, t \qquad a.s.$$

Proof. Decompose h as $h = \alpha k + j$, where $j \perp k$ and $(h, k) = \alpha(k, k)$. Then β^h is indistinguishable from $\alpha \beta^k + \beta^j$. Since the pair $\{\beta^j, \beta^k\}$ is a Gaussian process and

$$E(\beta^j(t) \, \beta^k(\tau)) = t \wedge \tau (j, k) = 0,$$

it follows that β^j and β^k are independent Brownian motions. Hence

$$\prec\beta^h, \beta^k\succ_t = \alpha\prec\beta^k, \beta^k\succ_t + \prec\beta^j, \beta^k\succ_t = \alpha(k, k)\, t + 0 = (h, k)\, t. \qquad \text{Q.E.D.}$$

COROLLARY 3.5. *Let $u, v \in [0, 1]$ and $A, B \in \mathfrak{g}$, then*

$$\prec\langle A, \beta(\cdot, u)\rangle, \langle B, \beta(\cdot, v)\rangle\succ_t = t\langle A, B\rangle\, G_0(u, v). \tag{3.4}$$

Proof. Take $h = G_0(\cdot, u)\, A$ and $k = G_0(\cdot, u)\, B$ in Lemma 3.4 and use $(h, k) = \langle A, B\rangle\, G_0(u, v)$. Q.E.D.

3.2. $\mathscr{L}(G)$-*Valued Brownian Motion*

Notation 3.6. Given an $\mathscr{L}$-valued process $\{\Sigma(t)\}_{t\geqslant 0}$ on $\mathscr{W}$, let $\Sigma(t, s)(\omega) \equiv \Sigma(t)(\omega)(s)$. In this way we will identify $\mathscr{L}$-valued processes on $\mathscr{W}$ with two parameter G-valued processes.

In preparation for proving the existence of a "Brownian Motion" on $\mathscr{L}(G)$, we will introduce a metric on G.

DEFINITION 3.7. The distance metric $d\colon G \times G \to G$ is defined by

$$d(g, h) = \inf \int_0^1 |\sigma'(s)|\, ds,$$

where the infimum is taken over all C^1-paths σ in G such that $\sigma(0) = g$ and $\sigma(1) = h$. Also set

$$|g| \doteq d(g, e) \qquad \forall g \in G.$$

Notice that

$$d(xg, xh) = d(g, h)$$

for all $g, h, x \in G$. Indeed, if σ is a curve joining g to h, then $x\sigma(\cdot)$ is a curve joining xg to xh which has the same length as σ. Set $|g| \equiv d(g, e) = d(e, g)$, then because of the above displayed equation,

$$d(g, h) = |g^{-1}h| = |h^{-1}g|.$$

Setting $h = e$ in this equation shows that $|g| = |g^{-1}|$ for all $g \in G$.

The next theorem is stated in Malliavin [17]. For the readers convenience we will supply a proof.

THEOREM 3.8 (Malliavin). *Suppose that G is a compact Lie group and $\langle \cdot, \cdot \rangle$ is an Ad_G invariant inner product on $\mathfrak{g}$. There exists a continuous adapted process $\{\Sigma(t)\}_{t \geqslant 0}$ on the filtered probability space $(\mathcal{W}, \{\mathcal{F}_t\}_{t \geqslant 0}, \mathcal{F}, P)$ such that for each $s \in [0, 1]$, $\Sigma(\cdot, s)$ solves the stochastic differential equation*:

$$\Sigma(\delta t, s) = \Sigma(t, s)\, \beta(\delta t, s) \qquad \text{with} \quad \Sigma(0, s) = g_0(s). \tag{3.5}$$

More precisely, Eq. (3.5) is shorthand notation for the stochastic differential equation

$$\Sigma(\delta t, s) = \sum_{A \in \mathfrak{g}_0} \tilde{A}(\Sigma(t, s))\, \beta^A(\delta t, s) \qquad \text{with} \quad \Sigma(0, s) = g_0(s), \tag{3.6}$$

where $\mathfrak{g}_0 \subset \mathfrak{g}$ is an orthonormal basis of $\mathfrak{g}$, $\tilde{A}$ is the left invariant vector field on G such that $\tilde{A}(e) = A$, and $\beta^A(t, s) \equiv \langle A, \beta(t, s) \rangle$. Here $\beta^A(\delta t, s)$ denotes the Stratonovich differential of the process $t \to \beta^A(t, s)$. In the sequel, we will use "δ" for Stratonovich differential and "d" for the differential of a semi-martingale.

Before starting the proof of this theorem, let us recall the following easy lemma.

LEMMA 3.9. *Let M and N be two finite dimensional manifolds, $\{X_i\}_{i=1}^n$ and $\{Y_i\}_{i=1}^n$ be a collection of smooth vector fields on M and N respectively, and $b(t) = (b_1(t), b_2(t), ..., b_n(t))$ (for $t \geqslant 0$) be an $\mathbb{R}^n$-valued continuous semi-martingale. (As usual b is defined on a filtered probability space satisfying the usual hypothesis.) Suppose that x and y are semi-martingales on M and N which satisfy the stochastic differential equations,*

$$\delta x = \sum_{i=1}^n X_i(x)\, \delta b_i \qquad \text{and} \qquad \delta y = \sum_{i=1}^n Y_i(y)\, \delta b_i,$$

respectively. Then (x, y) is a $M \times N$-valued semi-martingale satisfying the stochastic differential equation

$$\delta(x, y) = \sum_{i=1}^n (\hat{X}_i(x, y) + \hat{Y}_i(x, y))\, \delta b_i,$$

where $\hat{X}_i$ and $\hat{Y}_i$ are the smooth vector fields on $M \times N$ defined by $\hat{X}_i(x, y) \equiv (X_i(x), 0_y)$ and $\hat{Y}_i(x, y) \equiv (0_x, Y_i(y))$. Here 0_x and 0_y denote the zero tangent vectors in $T_x M$ and $T_y M$ respectively.

Proof. Let z be the $M \times N$-valued semi-martingale which solves the stochastic differential equation,

$$\delta z = \sum_{i=1}^{n} (\hat{X}_i(z) + \hat{Y}_i(z))\, \delta b_i \qquad \text{with} \quad z(0) = (x(0), y(0)). \tag{3.7}$$

To finish the proof it suffices to show that $z = (x, y)$.

Define $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ to be projections onto the first and second factor of $M \times N$ respectively and $z_i \equiv \pi_i(z)$. If $f \in C^\infty(M)$ then $f \circ \pi_1 \in C^\infty(M \times N)$. So by definition of z solving Eq. (3.7), we have

$$\delta(f \circ \pi_1(z)) = \sum_{i=1}^{n} [(\hat{X}_i(z) + \hat{Y}_i(z))(f \circ \pi_1)]\, \delta b_i.$$

Now $\hat{X}_i(z)(f \circ \pi_1) = (\pi_* \hat{X}_i(z))\, f = X_i(z_i)\, f$ and similarly $\hat{Y}_i(z)(f \circ \pi_1) = (\pi_* \hat{Y}_i(z))\, f = 0$. Therefore, the last displayed equation may be written as

$$\delta(f(z_1)) = \sum_{i=1}^{n} [X_i(z_1)\, f]\, \delta b_i,$$

i.e., z_1 is a semi-martingale on M solving the stochastic differential equation,

$$\delta z_1 = \sum_{i=1}^{n} X_i(z_1)\, \delta b_i \qquad \text{with} \quad z_1(0) = x(0).$$

Since this is the same equation solved by x it follows by uniqueness of solutions that $z_1 = x$. The same argument also shows that $z_2 = y$. That is $(x, y) = z$. Q.E.D.

Proof of Theorem 3.8. For the purposes of the proof we will adopt the following notation. If $f \in C^\infty(G)$, let $f' \in C^\infty(G, \mathfrak{g}^*)$ be defined by

$$f'(g)\langle A \rangle = df \langle \tilde{A}(g) \rangle = \frac{d}{dt}\Big|_0 f(ge^{tA})$$

and let

$$\Delta_G f \equiv \sum_{A \in \mathfrak{g}_0} \tilde{A}^2 f.$$

We also start with the special case where $g_0(s) \equiv e$ for all $s \in [0, 1]$. For each $s \in [0, 1]$, let $\Sigma^0(\cdot, s)$ be a solution to the stochastic differential equation in Eq. (3.5) (or equivalently Eq. (3.6)) with initial condition $\Sigma_0(0, s) = g_0(s) \equiv e \in G$. (For the existence of solutions to this equation see for example [8, 13, 14].) In this way we construct a G-valued two parameter process $\Sigma^0(t, s)$. Our immediate goal is to show that there exists a continuous version $(\Sigma(t, s))$ of this process.

For the moment, fix $\tau \in [0, \infty)$ and let $u(t) \equiv \Sigma^0(\tau, s)^{-1} \Sigma^0(t, s)$ for $t \geqslant \tau$. Then $u(\tau) = e \in G$ and u solves the stochastic differential equation,

$$\begin{aligned} \delta u(t) &= L_{\Sigma^0(\tau, s)^{-1} *} \sum_{A \in \mathfrak{g}_0} \tilde{A}(\Sigma^0(t, s))\, \beta^A(\delta t, s) \\ &= \sum_{A \in \mathfrak{g}_0} \tilde{A}(\Sigma^0(\tau, s)^{-1} \Sigma^0(t, s))\, \beta^A(\delta t, s) \\ &= \sum_{A \in \mathfrak{g}_0} \tilde{A}(u(t))\, \beta^A(\delta t, s), \end{aligned}$$

wherein we have used the left invariance of $\tilde{A}$ along with Eq. (3.6). So if $f \in C^\infty(G)$ such that $f(e) = 0$, then using Corollary 3.5 we find for all $t \geqslant \tau$ that

$$\begin{aligned} f(u(t)) &= f(u(\tau)) + \int_\tau^t \sum_{A \in \mathfrak{g}_0} (\tilde{A} f)(u(r))\, \beta^A(\delta r, s) \\ &= \int_\tau^t \sum_{A \in \mathfrak{g}_0} (\tilde{A} f)(u(r))\, \beta^A(dr, s) + \frac{1}{2} \int_\tau^t \sum_{A \in \mathfrak{g}_0} (\tilde{A}^2 f)(u(r))\, G_0(s, s)\, dr \\ &= \int_\tau^t \sum_{A \in \mathfrak{g}_0} (\tilde{A} f)(u(r))\, \beta^A(dr, s) + \frac{G_0(s, s)}{2} \int_\tau^t (\Delta f)(u(r))\, dr. \end{aligned}$$

For any $p \in [2, \infty)$ and $t \geqslant \tau$, it follows from Burkholder's inequality that

$$\begin{aligned} E\, |f(u(t))|^p &\leqslant C_p(f)\, E \left\{ \int_\tau^t |f'(u(r))|^2\, G_0(s, s)\, dr \right\}^{p/2} \\ &\quad + C_p(f)\, E \left| \int_\tau^t \Delta_G f(u(r))\, G_0(s, s)\, dr \right|^p \\ &\leqslant C_p(f) \{ (t - \tau)^{p/2} + (t - \tau)^p \}, \end{aligned} \tag{3.8}$$

where $C_p(f)$ denotes a constant depending only on p and bounds on f' and $\Delta_G f$. Let $\{f_i\}_{i=1}^n \subset C^\infty(G)$ be a suitable collection of functions such that $f_i(e)=0$ for all i and

$$|g| := d(g, e) \leqslant \sum_{i=1}^n |f_i(g)| \qquad \forall g \in G,$$

cf., the Whitney imbedding theorem. This equation and Eq. (3.8) implies, for all $t \geqslant \tau$ and all $s \in [0, 1]$, that

$$\begin{aligned} E[d(\Sigma^0(t, s), \Sigma^0(\tau, s))]^p &= E\,|\Sigma^0(\tau, s)^{-1}\,\Sigma^0(t, s)|^p = E\,|u(t)|^p \\ &\leqslant C_p\{(t-\tau)^{p/2} + (t-\tau)^p\}, \end{aligned}$$

where $C_p \equiv n^{(p-1)} \sum_{i=1}^n C_p(f_i)$. So we have proven for all $s \in [0, 1]$ and $t, \tau \in [0, \infty)$ that

$$E\{d(\Sigma^0(\tau, s), \Sigma^0(t, s))\}^p \leqslant C_p\{|t-\tau|^{p/2} + |t-\tau|^p\}. \tag{3.9}$$

Now fix $s, \sigma \in [0, 1]$, and set $u(t) \equiv \Sigma^0(t, s)\,\Sigma^0(t, \sigma)^{-1}$. In the case that G is a matrix group, we may compute $du(t)$ to find

$$\begin{aligned} du(t) &= \Sigma^0(t, s)\{\beta(\delta t, s) - \beta(\delta t, \sigma)\}\;\Sigma^0(t, \sigma)^{-1} \\ &= u(t)\,B(\delta t), \end{aligned}$$

where

$$B(t) = \int_0^t Ad_{\Sigma^0(\tau, \sigma)}\{\beta(\delta\tau, s) - \beta(\delta\tau, \sigma)\}. \tag{3.10}$$

To prove the analogous formula in the general case we will use Lemma 3.9. To this end let $f \in C^\infty(G)$ and $F(g, k) \equiv f(gk^{-1})$ so that $f(u(t)) = F(\Sigma^0(t, s), \Sigma^0(t, \sigma))$. For $A \in \mathfrak{g}$, we have

$$\begin{aligned} \frac{d}{d\varepsilon}\bigg|_0 F(ge^{\varepsilon A}, k) &= \frac{d}{d\varepsilon}\bigg|_0 f(gk^{-1}ke^{\varepsilon A}k^{-1}) = \frac{d}{d\varepsilon}\bigg|_0 f(gk^{-1}e^{\varepsilon Ad_k A}) \\ &= ((Ad_k \tilde{A})\,f)(gk^{-1}) \end{aligned}$$

and, by essentially the same computation,

$$\frac{d}{d\varepsilon}\bigg|_0 F(g, ke^{\varepsilon A}) = -((Ad_k \tilde{A})\,f)(gk^{-1}).$$

therefore,

$$d[f(u(t))] = \sum_{A \in \mathfrak{g}_0} ((Ad_{\Sigma^0(t,\sigma)}\tilde{A})f)(u(t)) \cdot (\beta^A(\delta t, s) - \beta^A(\delta t, \sigma))$$
$$= \sum_{A,\, C \in \mathfrak{g}_0} \langle Ad_{\Sigma^0(t,\sigma)}A, C\rangle (\tilde{C}f)(u(t)) \cdot \langle \beta(\delta t, s) - \beta(\delta t, \sigma), A\rangle$$
$$= \sum_{C \in \mathfrak{g}_0} (\tilde{C}f)(u(t)) \cdot \langle Ad_{\Sigma^0(t,\sigma)}(\beta(\delta t, s) - \beta(\delta t, \sigma)), C\rangle$$
$$= \sum_{C \in \mathfrak{g}_0} (\tilde{C}f)(u(t))\, \delta B^C(t),$$

where B is the process defined in Eq. (3.10).

We now claim that B may be expressed as

$$B(t) = \int_0^t Ad_{\Sigma^0(\tau,\sigma)}\{\beta(d\tau, s) - \beta(d\tau, \sigma)\}. \tag{3.11}$$

The main point here is that

$$d_t(Ad_{\Sigma^0(t,\sigma)}) = \sum_{A \in \mathfrak{g}_0} (\tilde{A}Ad_{(\cdot)})(\Sigma^0(t,\sigma))\, \delta\beta^A(t,\sigma)$$
$$= \sum_{A \in \mathfrak{g}_0} Ad_{\Sigma^0(t,\sigma)} ad_A\, \delta\beta^A(t,\sigma).$$

Using this equation we find

$$B(t) = \int_0^t Ad_{\Sigma^0(\tau,\sigma)}\{\beta(d\tau, s) - \beta(d\tau, \sigma)\}$$
$$+ \tfrac{1}{2} \int_0^t \sum_{A \in \mathfrak{g}_0} Ad_{\Sigma^0(t,\sigma)} ad_A\{\beta(d\tau, s) - \beta(d\tau, \sigma)\}\ \beta^A(d\tau, \sigma)$$

from which Eq. (3.11) follows because

$$\sum_{A \in \mathfrak{g}_0} ad_A\{\beta(d\tau, s) - \beta(d\tau, \sigma)\}\ \beta^A(d\tau, \sigma)$$
$$= \sum_{A,\, C \in \mathfrak{g}_0} ad_A C \cdot d_\tau \prec \{\beta^C(\cdot, s) - \beta^C(\cdot, \sigma)\}, \beta^A(\cdot, \sigma) \succ_\tau$$
$$= \sum_{A \in \mathfrak{g}_0} ad_A A \cdot (G_0(s,\sigma) - G_0(\sigma,\sigma))\, d\tau = 0.$$

Since $Ad_g\colon \mathfrak{g}\to\mathfrak{g}$ acts isometrically for all $g\in G$, we conclude from Eq. (3.11) that $B(t)$ is again a Brownian motion with the same covariance as the Brownian motion $t\to\beta(t,s)-\beta(t,\sigma)$. This covariance is

$$E\{(\beta(t,s)-\beta(t,\sigma))\otimes(\beta(t,s)-\beta(t,\sigma))\}\equiv tF(s,\sigma)\,\mathscr{I},$$

where $\mathscr{I}\equiv\sum_{A\in\mathfrak{g}_0}A\otimes A\in\mathfrak{g}\otimes\mathfrak{g}$ and $F(s,\sigma)\equiv G_0(s,s)+G_0(\sigma,\sigma)-2G_0(s,\sigma)$.

Notice for each fixed $\sigma\in[0,1]$ that $s\to F(s,\sigma)$ is a piecewise C^1-function such that $F(\sigma,\sigma)=0$ and

$$|\partial F(s,\sigma)/\partial s|=|1-2s-2(1_{s\leqslant\sigma}-\sigma)|\leqslant 4.$$

Therefore

$$F(s,\sigma)\leqslant 4\,|s-\sigma|,\qquad \forall s,\sigma\in[0,1].$$

By computations similar to those which lead to Eq. (3.8), if $f\in C^\infty(G)$ and $f(e)=0$, then

$$\begin{aligned}E\,|f(u(t))|^p&\leqslant C_pE\left\{\int_0^t|f'(u(\tau))|^2\,F(s,\sigma)\,d\tau\right\}^{p/2}\\&\quad+C_pE\left|\int_0^t\Delta_Gf(u(\tau))\,F(s,\sigma)\,d\tau\right|^p\\&\leqslant C_p(f)\{(tF(s,\sigma))^{p/2}+(tF(s,\sigma))^p\},\end{aligned}$$

where $C_p(f)$ denotes a constant depending only on p and the bounds on f' and Δ_Gf as above. As in the proof of Eq. (3.9), we may conclude

$$\begin{aligned}E\{d(\Sigma^0(t,s),\Sigma^0(t,\sigma))\}^p&=E\,|\Sigma^0(t,s)^{-1}\,\Sigma^0(t,\sigma)|^p=E\,|u(t)|^p\\&\leqslant C_p\{(tF(s,\sigma))^{p/2}+(tF(s,\sigma))^p\}\\&\leqslant C_p\{(t\,|s-\sigma|)^{p/2}+(t\,|s-\sigma|)^p\},\end{aligned}\tag{3.12}$$

where C_p is a constant only depending on p and the compact group G.

The triangle inequality and the estimates in (3.9) and (3.12) yields

$$\begin{aligned}&E\{d(\Sigma^0(t,s),\Sigma^0(\tau,\sigma))\}^p\\&\quad\leqslant C_p\{t\,|s-\sigma|^{p/2}+(t\,|s-\sigma|)^p+|t-\tau|^{p/2}+|t-\tau|^p\},\end{aligned}$$

where C_p is a constant dependig only on p and the compact group G. Consequently, for each $T\in(0,\infty)$, there is a constant $C_p(T)$ such that

$$E\{d(\Sigma^0(t,s),\Sigma^0(\tau,\sigma))\}^p\leqslant C_p(T)\{|s-\sigma|^{p/2}+|t-\tau|^{p/2}\},$$

for all $s, \sigma \in [0, 1]$ and $t, \tau \in [0, T]$. Hence, by Kolmogorov's continuity criteria (see for example Theorem 1.4.4 of Kunita [14] and Theorem 53, Chapter 4 of Protter [22]) there is a continuous version $(\Sigma(t, s))$ of $\Sigma^0(t, s)$ such that for all $\beta \in (0, 1/2)$ there exists a positive random variable (K_β) on $\mathscr{W}$ such that

$$d(\Sigma(t, s), \Sigma(\tau, \sigma)) \leqslant K_\beta\{|t-\tau|^\beta + |s-\sigma|^\beta\} \qquad \text{a.s.} \tag{3.13}$$

Furthermore, $EK_\beta^p < \infty$ for all $p \in (1, \infty)$. Since, for each $s \in [0, 1]$, $\Sigma(\cdot, s)$ is a version of $\Sigma^0(\cdot, s)$, it follows that Σ satisfies all the hypothesis of the theorem when $g_0(s) = e$.

For the general case let Σ be as in the special case just proved and define $\hat{\Sigma}(t, s) \equiv g_0(s)\, \Sigma(t, s)$. Then $\{\hat{\Sigma}(t)\}_{t \geqslant 0}$ is a continuous adapted $\mathscr{L}$-valued process satisfying the differential equation in Eq. (3.5). Q.E.D.

3.3. *Generator of the Process Σ*

The next theorem shows that the $\mathscr{L}(G)$-valued process $\Sigma(t)$ constructed above satisfies the standard martingale criteria of a Brownian motion. For this reason $\Sigma(t)$ deserves to be called an $\mathscr{L}(G)$-valued Brownian motion.

THEOREM 3.10. *Let $\mathscr{P} = \{0 < s_1 < s_2 < \cdots < s_n < 1\}$ be a partition on $[0, 1]$, $G^{\mathscr{P}}$ be the set of functions from $\mathscr{P}$ to G, $\Sigma(t)$ be the $\mathscr{L}$-valued process in Theorem 3.8 and $f: [0, T] \times \mathscr{L} \to \mathbb{R}$ be a function of the form $f(t, g) = F(t, g_{\mathscr{P}})$. Assume that $F: [0, T] \times G^{\mathscr{P}} \to \mathbb{R}$ is a continuous function satisfying:*

1. *$F|_{(0, T) \times G^{\mathscr{P}}}$ is smooth and*

2. *The derivatives of $F|_{(0, T) \times G^{\mathscr{P}}}$ up to second order extend to continuous functions on $[0, T] \times G^{\mathscr{P}}$.*

Then

$$f(t, \Sigma(t)) = f(0, \Sigma(0)) + M_t + \int_0^t \left(\left(\frac{\partial}{\partial \tau} + \frac{1}{2}\Delta\right) f(\tau, \cdot)\right)(\Sigma(\tau))\, d\tau,$$

where M_t is the martingale:

$$\begin{aligned} M_t &= \sum_{i=1}^n \sum_{A \in \mathfrak{g}_0} \int_0^t (A^{(i)}F)(\tau, \Sigma_{\mathscr{P}}(\tau)) \langle \beta(d\tau, s_i), A\rangle \\ &= \int_0^t ((\vec{\nabla} f)(\tau, \Sigma(\tau)), \beta(d\tau)). \end{aligned}$$

Proof. Set

$$\Sigma_{\mathscr{P}}(t) \equiv (\Sigma(t)_{\mathscr{P}}) = (\Sigma(t, s_1), ..., \Sigma(t, s_n)).$$

Then, by Lemma 3.9, $\Sigma_{\mathscr{P}}(t) \in G^n$ satisfies the stochastic differential equation

$$\delta\Sigma_{\mathscr{P}}(t) = \sum_{i=1}^{n} \sum_{A \in \mathfrak{g}_0} A^{(i)}(\Sigma_{\mathscr{P}}(t)) \langle \beta(\delta t, s_i), A \rangle, \tag{3.14}$$

where for $A \in \mathfrak{g}$, $A^{(i)}$ is defined in Eq. (2.5).

Equations (3.4) and (2.3) allows us to compute, using Itô's lemma in finite dimensions, the differential of $f(t, \Sigma(t))$ as

$$\begin{aligned} d[f(t, \Sigma(t))] &= \partial f(t))/\partial t + \sum_{i=1}^{n} \sum_{A \in \mathfrak{g}_0} (A^{(i)}F)(t, \Sigma_{\mathscr{P}}(t)) \langle \beta(\delta t, s_i), A \rangle \\ &= \partial f(t, \Sigma(t))/\partial t + \sum_{i=1}^{n} \sum_{A \in \mathfrak{g}_0} (A^{(i)}F)(t, \Sigma_{\mathscr{P}}(t)) \langle \beta(dt, s_i), A \rangle \\ &\quad + \tfrac{1}{2} \sum_{i=1}^{n} \sum_{A \in \mathfrak{g}_0} \sum_{j=1}^{n} \sum_{B \in \mathfrak{g}_0} (B^{(j)}A^{(i)}F)(t, \Sigma_{\mathscr{P}}(t)) \langle \beta(dt, s_i), A \rangle \langle \beta(dt, s_j), B \rangle \\ &= dM_t + \partial f(t, \Sigma(t))/\partial t + \tfrac{1}{2} \sum_{i=1}^{n} \sum_{A \in \mathfrak{g}_0} \sum_{j=1}^{n} (A^{(j)}A^{(i)}F)(t, \Sigma_{\mathscr{P}}(t))\, G_0(s_i, s_j)\, dt \\ &= dM_t + \partial f(t, \Sigma(t))/\partial t + \tfrac{1}{2}(\Delta f)(t, \Sigma(t))\, dt. \end{aligned}$$

In the above computation we have used $\langle \beta(t, s), A \rangle = \beta^{G_0(s, \cdot)A}(t)$, see the proof of Corollary 3.5, and Eq. (2.4) to conclude that

$$\sum_{i=1}^{n} \sum_{A \in \mathfrak{g}_0} (A^{(i)}F)(t, \Sigma_{\mathscr{P}}(t)) \langle \beta(dt, s_i), A \rangle = (\vec{\nabla} f(t, \Sigma(t)), d\beta(t)). \qquad \text{Q.E.D.}$$

Notation 3.11. For definiteness, in the remainder of this paper let $\{\Sigma(t)\}_{t \geqslant 0}$ denote the $\mathscr{L}(G)$ valued process constructed in Theorem 3.8 with $g_0(s) \equiv e$ for all $s \in [0, 1]$.

By the proof of Theorem 3.10, if $F \in C^\infty(G^{\mathscr{P}})$ then

$$dF(\Sigma_{\mathscr{P}}(t)) = \sum_{i=1}^{n} \sum_{A \in \mathfrak{g}_0} (A^{(i)}F)(\Sigma_{\mathscr{P}}(t)) \langle \beta(dt, s_i), A \rangle$$
$$+ \tfrac{1}{2} \sum_{A \in \mathfrak{g}_0} \sum_{i,j=1}^{n} (A^{(j)}A^{(i)}F)(\Sigma_{\mathscr{P}}(t))\, G_0(s_i, s_j)\, dt.$$

Thus $\Sigma_{\mathscr{P}}(t)$ is a diffusion process on $G^{\mathscr{P}}$ with generator

$$\Delta_{\mathscr{P}} \equiv \tfrac{1}{2} \sum_{A \in \mathfrak{g}_0} \sum_{i,j=1}^{n} G_0(s_i, s_j)\, A^{(j)}A^{(i)}. \tag{3.15}$$

LEMMA 3.12. *Let* $\mathscr{P} = \{0 < s_1 < s_2 < \cdots < s_n < 1\}$ *be a partition of* $[0, 1]$, *then the matrix* $\{G_0(s_i, s_j)\}_{i,j=1}^{n}$ *is positive definite. In particular* $\Delta_{\mathscr{P}}$ *is a second order elliptic differential operator on* $G^{\mathscr{P}}$.

Proof. Let $H_0(\mathbb{R})$ be the set of absolutely continuous functions $l: [0, 1] \to \mathbb{R}$ such that $l(1) = l(0) = 0$ and $(l, l) \equiv \int_0^1 (l'(s))^2\, ds < \infty$. Choose an orthonormal basis $\mathfrak{h}$ of $H_0(\mathbb{R})$. Then by Lemma 3.8 of [6], $G_0(s, \sigma) = \sum_{l \in \mathfrak{h}} l(s)\, l(\sigma)$ with the sum being absolutely convergent therefore, if $\xi = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$,

$$\sum_{i,j=1}^{n-1} G_0(s_i, s_j)\, \xi_i \xi_j = \sum_{i,j=1}^{n-1} \sum_{l \in \mathfrak{h}} l(s_i)\, l(s_j)\, \xi_i \xi_j = \sum_{l \in \mathfrak{h}} \left[\sum_{i=1}^{n-1} l(s_i)\, \xi_i \right]^2 \geqslant 0$$

with equality iff $\sum_{i=1}^{n-1} l(s_i)\, \xi_i = 0$ for all $l \in \mathfrak{h}$. Since $\mathfrak{h}$ is an orthonormal basis for $H_0(\mathbb{R})$ and the map $l \in H_0(\mathbb{R}) \to l(s) \in \mathbb{R}$ is a continuous linear functional, the condition $\sum_{i=1}^{n-1} l(s_i)\, \xi_i = 0$ for all $l \in \mathfrak{h}$, is equivalent to $\sum_{i=1}^{n-1} l(s_i)\, \xi_i = 0$ for all $l \in H_0(\mathbb{R})$. Choosing $l \in H_0(\mathbb{R})$ such that l is nonzero on exactly one of the partition points in $\mathscr{P}$ allows us to conclude that $\sum_{i,j=1}^{n-1} G_0(s_i, s_j)\, \xi_i \xi_j = 0$ iff $\xi = 0$. Q.E.D.

Before ending this section let us record a slight extension of Theorem 3.10 which will be needed in the sequel. (The proof will be left to the reader.) We will first need the following definition.

DEFINITION 3.13. Given a Hilbert space T, let ∇^0 denote the "flat" covariant derivative on $\mathscr{F}C^\infty(\mathscr{L}, T)$ defined by $\nabla_h^0 f \equiv \tilde{h} f$ for all $f \in \mathscr{F}C^\infty(\mathscr{L}, T)$ and $h \in H_0(\mathfrak{g})$. Let Δ^0 denote the "flat" Laplacian on $\mathscr{F}C^\infty(\mathscr{L}, T)$ defined by

$$\Delta^0 f \equiv \sum_{h \in S_0} (\nabla_h^0)^2 f = \sum_{h \in S_0} \tilde{h}^2 f.$$

THEOREM 3.14. *Suppose that* T *is a Hilbert space and* $f: [0, T] \times \mathscr{L}(G) \to T$ *is a function of the form* $f = \sum_{i=1}^n f_i \xi_i$ *where* $f_i: [0, T] \times \mathscr{L}(G) \to T$ *are functions satisfying the assumptions in Theorem* 3.10 *and* $\xi_i \in T$ *for* $i = 1, 2, ..., n$. *Then*

$$d(f(t, \Sigma(t))) = (\nabla^0_{d\beta(t)} f)(\Sigma(t)) + \left(\left(\frac{\partial}{\partial t} + \frac{1}{2}\Delta^0\right) f\right)(t, \Sigma(t))\, dt. \quad (3.16)$$

4. INTEGRATION BY PARTS ON THE PATH SPACE OF $\mathscr{L}(G)$

In this section, integration by parts formulas on the path space of the loop group are derived. As a corollary we will show that the pre-Dirichlet form introduced in Driver and Lohrenz [6] is closable. Before doing this however, it is first necessary to discuss parallel translation ($//_t$) along the Brownian motion $\Sigma(t)$. Pretending for the moment that $\mathscr{L}(G)$ is a finite dimensional Lie group, $//_t k_0 = L_{\Sigma(t)*} k(t)$, where $k(t)$ is the solution to the stochastic differential equation,

$$dk(t) + D_{\delta\beta(t)} k(t) = 0 \qquad \text{with} \quad k(0) = k_0, \quad (4.1)$$

where $\delta\beta$ denotes the Stratonovich differential of β. For motivation, see Theorem 6.3 in [5]. Formally, writting "$\beta = \sum_{h \in S_0} \beta^h h$,"

$$\begin{aligned} D_{\delta\beta} k &= D_{d\beta} k + \tfrac{1}{2} D_{d\beta}\, dk = D_{d\beta} k - \tfrac{1}{2} D_{d\beta} D_{d\beta} k \\ &= D_{d\beta} k - \tfrac{1}{2} \sum_{h \in S_0} D_h D_h k\, dt \\ &= D_{d\beta} k - \tfrac{1}{2} \Delta^{(1)} k\, dt. \end{aligned}$$

Hence we should interpret Eq. (4.1) as the Itô equation,

$$dk(t) = -D_{d\beta(t)} k(t) + \tfrac{1}{2} \Delta^{(1)} k(t)\, dt \qquad \text{with} \quad k(0) = k_0. \quad (4.2)$$

See the Appendix (Section 8) for a review of the Itô integral in this context.

4.1. *Parallel Translation*

THEOREM 4.1 (Parallel Translation). *For each* $k_0 \in H_0(\mathfrak{g})$, *there exists a unique solution to Eq.* (4.2). *Moreover if* $h_0 \in H_0(\mathfrak{g})$ *and* h *is the solution to Eq.* (4.2) *with* k *and* k_0 *replaced by* h *and* h_0 *respectively, then for all* $t \geq 0$, $(h(t), k(t)) = (h_0, k_0)$ *alomost surely.*

Proof. Eq. (4.2) may be solved using the usual Picard iterates scheme. Recall that $\Delta^{(1)} = \Delta|_{H_0(\mathfrak{g})}$ is a bounded operator and notice that

$$E\left\| \int_0^t D_{\beta(d\tau)} k(\tau) \right\|^2 = E \int_0^t \sum_{h \in S_0} \|D_h k\|^2 \, d\tau \leqslant \|D\|_{op}^2 \, E \int_0^t \|k\|^2 \, d\tau$$

when $\{k(t)\}_{t \geqslant 0}$ is a continuous adapted $H_0(\mathfrak{g})$-valued process such that $E \int_0^t \|k\|^2 \, d\tau < \infty$.

To simplify notation, if f is a possibly random function on $[0, \infty)$ taking values in a normed space T, let $f^*(t) \equiv \sup\{\|f(\tau)\| : 0 \leqslant \tau \leqslant t\}$. If

$$k_n(t) \equiv k_0 - \int_0^t D_{\beta(d\tau)} k_{n-1}(\tau) + \tfrac{1}{2} \int_0^t \Delta^{(1)} k_{n-1}(\tau) \, d\tau, \tag{4.3}$$

then

$$\begin{aligned} k_{n+1}(t) - k_n(t) = {} & -\int_0^t D_{\beta(d\tau)}(k_n(\tau) - k_{n-1}(\tau)) \\ & + \tfrac{1}{2} \int_0^t \Delta^{(1)}(k_n(\tau) - k_{n-1}(\tau)) \, d\tau. \end{aligned}$$

Hence using Burkholder's inequality

$$\begin{aligned} E(k_{n+1} - k_n)^{*2}(t) \leqslant {} & 2E\left(\int_0^{\cdot} D_{\beta(d\tau)}(k_n(\tau) - k_{n-1}(\tau))\right)^{*2}(t) \\ & + \frac{t}{2} \|\Delta^{(1)}\|_{op}^2 \int_0^t E(k_n - k_{n-1})^{*2}(\tau) \, d\tau \\ \leqslant {} & 4 \|D\|_{op}^2 \int_0^t E\|k_n(\tau) - k_{n-1}(\tau)\|^2 \, d\tau \\ & + \frac{t}{2} \|\Delta^{(1)}\|_{op}^2 \int_0^t E(k_n - k_{n-1})^{*2}(\tau) \, d\tau \\ \leqslant {} & 4 \|D\|_{op}^2 \int_0^t E(k_n - k_{n-1})^{*2}(\tau) \, d\tau \\ & + \frac{t}{2} \|\Delta^{(1)}\|_{op}^2 \int_0^t E(k_n - k_{n-1})^{*2}(\tau) \, d\tau \\ = {} & \left(4 \|D\|_{op}^2 + \frac{t}{2} \|\Delta^{(1)}\|_{op}^2\right) \int_0^t E(k_n - k_{n-1})^{*2}(\tau) \, d\tau. \end{aligned}$$

Fix $T>0$ and let $K_T \equiv (4\,\|D\|^2_{op} + (T/2)\,\|\Delta^{(1)}\|^2_{op})$, and $f_n(t) = E(k_n - k_{n-1})^{*2}(t)$. Then for $0 \leqslant t \leqslant T$,

$$f_{n+1}(t) \leqslant K_T \int_0^t f_n(\tau)\,d\tau$$

which implies after iteration that

$$f_n(t) \leqslant \frac{(K_T t)^n}{n!} f_0^*(T).$$

Thus $\sum_{n=0}^{\infty} f_n^*(T) \leqslant f_0^*(T) \cdot \exp(K_T T)$, from which we learn that $k_n(t)$ is L^2-uniformly convergent for t in compact subsets of $[0, \infty)$ to a continuous process, say $k(t)$. Passing to the limit in Eq. (4.3) shows that k solves,

$$k(t) \equiv k_0 - \int_0^t D_{\beta(d\tau)} k(\tau) + \tfrac{1}{2} \int_0^t \Delta^{(1)} k(\tau)\,d\tau. \tag{4.4}$$

Let k be as above and $h(t)$ be a solution to Eq. (4.2) (or equivalently (4.4)) with $h(0) = h_0$ and set $F_t l \equiv -D_l h(t)$ and $G_t l \equiv -D_l k(t)$ for all $l \in H_0(\mathfrak{g})$. Then

$$dh(t) = F_t\,d\beta(t) + \tfrac{1}{2}\,\Delta^{(1)} h(t)\,dt$$

and

$$dk(t) = G_t\,d\beta(t) + \tfrac{1}{2}\,\Delta^{(1)} k(t)\,dt.$$

Therefore by Itô's Lemma, see Theorem 8.5 of the Appendix,

$$\begin{aligned} d(h(t), k(t)) = {} & (F_t^* k(t), d\beta(t)) + \tfrac{1}{2}(\Delta^{(1)} h(t)\,dt, k(t)) \\ & + (G_t^* h(t), d\beta(t)) + \tfrac{1}{2}(h(t), \Delta^{(1)} k(t)\,dt) \\ & + (F_t, G_t)_{HS}\,dt, \end{aligned}$$

where, for all $u, v \in HS(H_0(\mathfrak{g})) \cong H_0(\mathfrak{g})^* \otimes H_0(\mathfrak{g})$ (the Hilbert Schmidt operators on $H_0(\mathfrak{g})$) and

$$(u, v)_{HS} \equiv \sum_{h \in S_0} (uh, vh). \tag{4.5}$$

For all $l \in H_0(\mathfrak{g})$,

$$(F_t^* k(t), l) + (G_t^* h(t), l) = (k(t), F_t l) + (h(t), G_t l)$$
$$= (k(t), -D_l(t)) + (h(t), -D_l k(t)) = 0,$$

since D_l is skew adjoint. Also

$$(F_t, G_t)_{HS} = \sum_{l \in S_0} (F_t l, G_t l) = \sum_{l \in S_0} (D_l h(t), D_l k(t))$$
$$= \sum_{l \in S_0} (h(t), -D_l^2 k(t)) = -(h(t), \Delta^{(1)} k(t)).$$

Combining the last four equations and using $\Delta^{(1)}$ is self-adjoint shows that $d(h(t), k(t)) = 0$. Q.E.D.

DEFINITION 4.2. Let $O(H_0(\mathfrak{g}))$ be the group of unitary operators on $H_0(\mathfrak{g})$ and $U(t)$ be the $O(H_0(\mathfrak{g}))$-valued process defined by

$$U(t)\, h_0 \equiv \sum_{k_0 \in S_0} (k_0, h_0)\, k(t) \qquad \text{for all} \quad h_0 \in H_0(\mathfrak{g}), \tag{4.6}$$

where for each $k_0 \in S_0$, $k(t)$ is the solution to (4.2) with $k(0) = k_0$.

LEMMA 4.3. *Suppose that $h_0 \in H_0(\mathfrak{g})$ and $h(t) \equiv U(t)\, h_0$, then h is a solution to* (4.2) *with $h(0) = h_0$. Moreover, $t \to U(t)$ is a.s. strongly continuous.*

Proof. Let $\{S_n\}$ be an increasing sequence of finite subsets of S_0 such that $\bigcup S_n = S_0$. Set

$$H_n(t) \equiv \sum_{k_0 \in S_n} (k_0, h_0)\, k(t),$$

so that

$$\|(U(t)\, h_0 - H_n(t)\|^2 = \left\| \sum_{k_0 \in S_0 \setminus S_n} (k_0, h_0)\, k(t) \right\|^2$$
$$= \sum_{k_0 \in S_0 \setminus S_n} |(k_0, h_0)|^2$$

which tends to zero uniformly in t as $n \to \infty$. This shows that $U(t)\, h_0$ is continuous, i.e., $U(t)$ is strongly continuous.

Let $h(t)$ denote the solution to (4.2) with $h(0)=h_0$. In order to prove that $U(t)\,h_0$ solves Eq. (4.2) with initial condition h_0, it suffices to show that $U(t)\,h_0=h(t)$ or equivalently that

$$\lim_{n\to\infty} E\,\|H_n(t)-h(t)\|^2=0.$$

Now it is clear that H_n solves (4.2) with initial condition $h_n\equiv\sum_{k_0\in S_n}(k_0,h_0)\,k_0$. Therefore we have

$$h(t)-H_n(t)=(h_0-h_n)-\int_0^t D_{\beta(d\tau)}(h(\tau)-H_n(\tau))$$
$$+\tfrac{1}{2}\int_0^t \Delta^{(1)}(h(\tau)-H_n(\tau))\,d\tau,$$

from which it follows that

$$E\,\|H_n(t)-h(t)\|^2\leqslant 3\,\|(h_0-h_n)\|^2+3E\int_0^t\sum_{l\in S_0}\|D_l(h(\tau)-H_n(\tau))\|^2\,d\tau$$
$$+\tfrac{3}{2}E\left|\int_0^t\|\Delta^{(1)}(h(\tau)-H_n(\tau))\|\,d\tau\right|^2$$
$$\leqslant 3\,\|(h_0-h_n)\|^2+3(\|D\|_{op}^2+\|\Delta^{(1)}\|^2\,t/2)$$
$$\times\int_0^t E\,\|(h(\tau)-H_n(\tau))\|^2\,d\tau.$$

It now follows by Gronwall's Lemma that $E\,\|H_n(t)-h(t)\|^2\to 0$ as $n\to\infty$.
Q.E.D.

THEOREM 4.4. *Suppose that $\dot{h}\colon[0,\infty)\times\mathscr{W}\to H_0(\mathfrak{g})$ is a progressively measurable process, i.e., $\dot{h}|_{[0,t]\times\mathscr{W}}$ is $\mathscr{B}([0,t])\otimes\mathscr{F}_t/\mathscr{B}(H_0(\mathfrak{g}))$-measurable for all $t\in(0,\infty)$. (Here $\mathscr{B}([0,t])$ and $\mathscr{B}(H_0(\mathfrak{g}))$ are the Borel σ-algebras on $[0,t]$ and $H_0(\mathfrak{g})$ respectively.) Also assume that*

$$E\left\{\int_0^t\|\dot{h}(\tau)\|^2\,d\tau\right\}<\infty\qquad\forall t>0\tag{4.7}$$

and set $h(t)=\int_0^t\dot{h}(\tau)\,d\tau$. Hence h is almost surely absolutely continuous with derivative given by $\dot{h}(t)$. Then

$$d(U(t)\,h(t))=-D_{d\beta(t)}(U(t)\,h(t))+\tfrac{1}{2}\Delta^{(1)}(U(t)\,h(t))\,dt+U(t)\,\dot{h}(t)\,dt\tag{4.8}$$

or equivalently, *because* $h(0)=0$,

$$U(t)\,h(t) = -\int_0^t D_{d\beta(\tau)}(U(\tau)\,h(\tau)) + \int_0^t \tfrac{1}{2}\Delta^{(1)}(U(\tau)\,h(\tau))\,d\tau$$
$$+\int_0^t U(\tau)\,\dot{h}(\tau)\,d\tau. \tag{4.9}$$

Proof. Let us first assume there is a constant $M<\infty$ such that almost surely, $\sup_{t\in[0,\infty)}\|\dot{h}(t)\|_{H_0(\mathfrak{g})} \leqslant M$. Let $\pi=\{0=t_0<t_1<t_2<\cdots<t_n=t\}\subset[0,t]$ be a partition of $[0,t]$ and $|\pi|\equiv\max\{|t_{i+1}-t_i| : i\in\{0,1,2,...,n-1\}\}$. For $\tau=t_i\in\pi$ let $\tau+\equiv t_{(i+1)\wedge n}$. Then

$$\begin{aligned}
U(t)\,h(t)-h(0) &= \sum_{\tau\in\pi}\{U(\tau+)\,h(\tau+)-U(\tau)\,h(\tau)\}\\
&= \sum_{\tau\in\pi}(U(\tau+)-U(\tau))\,h(\tau)+\sum_{\tau\in\pi}U(\tau)(h(\tau+)-h(\tau))\\
&\quad+\sum_{\tau\in\pi}(U(\tau+)-U(\tau))(h(\tau+)-h(\tau))\\
&=: A_\pi+B_\pi+C_\pi.
\end{aligned}$$

For $\xi\in(\tau,\tau+]$, let $h_\pi(\xi)=h(\tau)$, and $U_\pi(\xi)=U(\tau)$, $\delta_\pi(\xi)\equiv(U(\tau+)-U(\tau))$. With this notation we have

$$A_\pi = -\int_0^t D_{d\beta(\xi)}(U(\xi)\,h_\pi(\xi)) + \int_0^t \tfrac{1}{2}\Delta^{(1)}(U(\xi)\,h_\pi(\xi))\,d\xi,$$

$$B_\pi = \int_0^t U_\pi(\xi)\,\dot{h}(\xi)\,d\xi,$$

and

$$C_\pi = \int_0^t \delta_\pi(\xi)\,\dot{h}(\xi)\,d\xi.$$

If

$$A \equiv -\int_0^t D_{d\beta(\tau)}(U(\tau)\,h(\tau)) + \int_0^t \tfrac{1}{2}\Delta^{(1)}(U(\tau)\,h(\tau))\,d\tau$$

and

$$B \equiv \int_0^t U(\tau)\,\dot{h}(\tau)\,d\tau,$$

then

$$E\,\|A-A_\pi\|^2 \leqslant 2E\int_0^t \sum_{l\in S_0} \|D_l(U(\xi)(h(\xi)-h_\pi(\xi))\|^2\,d\xi$$

$$+2E\left(\int_0^t \tfrac{1}{2}\,\|\varDelta^{(1)}(U(\xi)(h(\xi)-h_\pi(\xi))\|\,d\xi\right)^2$$

$$\leqslant 2\,\|D\|_{op}^2\,E\int_0^t \|h(\xi)-h_\pi(\xi)\|^2\,d\xi + 2\,\|\varDelta^{(1)}\|_{op}^2$$

$$\times E\left(\int_0^t \|h(\xi)-h_\pi(\xi)\|\,d\xi\right)^2$$

which tends to zero as $|\pi|\to 0$ by the Dominated convergence theorem. Similarly

$$\|B-B_\pi\| \leqslant \int_0^t \|(U(\xi)-U_\pi(\xi))\,\dot h(\xi)\|\,d\xi.$$

Therefore, the strong continuity of U and the dominated convergence theorem implies that $B\to B_\pi$ as $|\pi|\to 0$ a.s. Finally the estimate

$$\|C_\pi\| \leqslant \int_0^t \|\delta_\pi(\xi)\,\dot h(\xi)\|\,d\xi,$$

the strong continuity of U, and the dominated convergence theorem implies that $\lim_{|\pi|\to 0} C_\pi = 0$. This proves Eq. (4.9) in the case that $\dot h$ is bounded.

For the general case, let $\dot h_n(t) \equiv 1_{[0,n]}(\|\dot h(\tau)\|)\,\dot h(\tau)$ and

$$h_n(t) \equiv \int_0^t 1_{[0,n]}(\|\dot h(\tau)\|)\,\dot h(\tau)\,d\tau.$$

Since $\|\dot h_n(t)\| \leqslant n$, we know that Eq. (4.9) holds with h replaced by h_n, i.e.,

$$U(t)\,h_n(t) = -\int_0^t D_{d\beta(\tau)}(U(\tau)\,h_n(\tau)) + \int_0^t \tfrac{1}{2}\,\varDelta^{(1)}(U(\tau)\,h_n(\tau))\,d\tau$$

$$+\int_0^t U(\tau)\,\dot h_n(\tau)\,d\tau. \tag{4.10}$$

By the Dominated convergence theorem,

$$\int_0^t \|\dot{h}(\tau)-\dot{h}_n(\tau)\|^2\,d\tau=\int_0^t |1-1_{[0,n]}(\|\dot{h}_n(\tau)\|)|^2\,\|\dot{h}(\tau)\|^2\,d\tau\to 0$$

$$\text{a.s.}\qquad n\to\infty. \tag{4.11}$$

We also have the Sobolev estimate,

$$\begin{aligned}\|h(t)-h_n(t)\|^2 &\leqslant\left(\int_0^t \|\dot{h}(\tau)-\dot{h}_n(\tau)\|\,d\tau\right)^2\\ &\leqslant t\int_0^t \|\dot{h}(\tau)-\dot{h}_n(\tau)\|^2\,d\tau.\end{aligned} \tag{4.12}$$

Using equations (4.11), (4.12), the facts that $\Delta^{(1)}$ is bounded and $U(t)$ is unitary, and the L^2-isometry property of the Itô integral, it is easy to let $n\to\infty$ in Eq. (4.10) to conclude that Eq. (4.9) holds for this general h as well. Q.E.D.

4.2. *Inegration by Parts*

LEMMA 4.5. *Let $G\in\mathscr{F}C^\infty(\mathscr{L},H_0(\mathfrak{g}))$ and $k\in H_0(\mathfrak{g})$, then*

$$\tfrac{1}{2}(\Delta G(g),k)=\tfrac{1}{2}\{(\Delta^0 G(g),k)+(G(g),\Delta^{(1)}k)\}-(\nabla^0 G(g),Dk)_{HS},$$

where $(\cdot,\cdot)_{HS}$ is defined in Eq. (4.5), *Δ in Definition* 2.3, *Δ_0 in Definition* 3.13 *and $\Delta^{(1)}=\Delta|_{H_0(\mathfrak{g})}$ in Theorem* 2.5.

Proof. *Using the skew symmetry properties of D_h we find*

$$\begin{aligned}(\Delta G(g),k)&=\sum_{h\in S_0}(\nabla_h^2 G(g),k)=\sum_{h\in S_0}((\tilde{h}+D_h)^2\,G(g),k)\\ &=(\Delta^0 G(g),k)+2\sum_{h\in S_0}(D_h\tilde{h}G(g),k)+\sum_{h\in S_0}(D_h^2\,G(g),k)\\ &=(\Delta^0 G(g),k)-2\sum_{h\in S_0}(\nabla_h^0 G(g),D_hk)+\sum_{h\in S_0}(G(g),D_h^2k)\\ &=(\Delta^0 G(g),k)-2(\nabla^0 G(g),Dk)_{HS}+(G(g),\Delta^{(1)}k).\end{aligned}$$

Q.E.D.

Notation 4.6. Suppose that $f\in\mathscr{F}C^\infty(\mathscr{L})$ of the form $f(g)=F(g_{\mathscr{P}})$, where $\mathscr{P}$ is a partition of $[0,1]$ and $F\colon G^{\mathscr{P}}\to\mathbb{R}$ is a smooth function. Define $(e^{t\Delta/2}f)\colon\mathscr{L}(G)\to\mathbb{R}$ by

$$(e^{t\Delta/2}f)(g)\equiv(e^{t\Delta_{\mathscr{P}}/2}F)(g_{\mathscr{P}}), \tag{4.13}$$

where $\Delta_{\mathscr{P}}$ is defined in Eq. (3.15) above.

Since $\Delta_{\mathscr{P}}$ is the generator of $\Sigma_{\mathscr{P}}(\cdot)$, we could also write Eq. (4.13) as

$$(e^{t\Delta/2}f)(g) \equiv E[F(g_{\mathscr{P}}\Sigma_{\mathscr{P}}(t))] = E[f(g\Sigma(t))], \tag{4.14}$$

where Σ solves Eq. (3.5) with $\Sigma(0, s) \equiv e$ for all $s \in [0, 1]$. It should also be noted that

$$\frac{\partial(e^{t\Delta/2}f)(g)}{\partial t} = \frac{1}{2}(\Delta_{\mathscr{P}} e^{t\Delta_{\mathscr{P}}/2}F)(g_{\mathscr{P}}) = \frac{1}{2}(\Delta e^{t\Delta/2}f)(g),$$

where we have used Eqs. (2.3) and (3.15) to conclude that

$$(\Delta_{\mathscr{P}} e^{t\Delta_{\mathscr{P}}/2}F)(g_{\mathscr{P}}) = (\Delta e^{t\Delta/2}f)(g). \tag{4.15}$$

PROPOSITION 4.7. *Let $h \in H(H_0(\mathfrak{g}))$, $f \in \mathscr{F}C^\infty(\mathscr{L})$, and $T > 0$. Set $F_t \equiv e^{(T-t)\Delta/2}f$ then*

$$\begin{aligned} d(\vec{\nabla}F_t(\Sigma(t)), U(t)\, h(t)) = {} & ((\nabla_{d\beta(t)} \vec{\nabla}F_t)(\Sigma(t)), U(t)\, h(t)) \\ & + (\vec{\nabla}F_t(\Sigma(t)), U(t)\, \dot{h}(t) + \tfrac{1}{2}\operatorname{Ric} U(t)\, h(t))\, dt. \end{aligned}$$

Proof. By Itô's Lemma, Theorem 3.14 above and the equalities $\partial\vec{\nabla}F_t/\partial t = \vec{\nabla}\Delta F_t/2$ and $\partial F_t/\partial t = \Delta F_t/2$, we have

$$d(\vec{\nabla}F_t(\Sigma(t))) = ((\nabla^0_{d\beta(t)} \vec{\nabla}F_t)(\Sigma(t)) + \tfrac{1}{2}(\Delta^0 \vec{\nabla}F_t(\Sigma(t)) - \vec{\nabla}\Delta F_t(\Sigma(t)))\, dt.$$

Using this equation, Theorem 4.4 and Itô's Lemma (see Theorem 3.14 above and Theorem 8.5 in the Appendix),

$$\begin{aligned} & d(\vec{\nabla}F_t(\Sigma(t)), U(t)\, h(t)) \\ & \quad = ((\nabla^0_{d\beta(t)} \vec{\nabla}F_t)(\Sigma(t)) + \tfrac{1}{2}(\Delta^0\, \vec{\nabla}F_t(\Sigma(t)) - \vec{\nabla}\, \Delta F_t(\Sigma(t)))\, dt, U(t)\, h(t)) \\ & \qquad + (\vec{\nabla}F_t(\Sigma(t)), -D_{d\beta(t)}(U(t)\, h(t)) + \tfrac{1}{2}\Delta(U(t)\, h(t))\, dt + U(t)\, \dot{h}(t)\, dt) \\ & \qquad + (\nabla^0\, \vec{\nabla}F_t(\Sigma(t)), -D(U(t)\, h(t)))_{HS}\, dt. \end{aligned}$$

The above expression may be simplified using $\nabla = \nabla^0 + D$ and Lemma 4.5 above to get

$$\begin{aligned} d(\vec{\nabla}F_t(\Sigma(t)), U(t)\, h(t)) = {} & ((\nabla d\beta(t)\, \vec{\nabla}F_t)(\Sigma(t)), U(t)\, h(t)) \\ & + (\vec{\nabla}F_t(\Sigma(t)), U(t)\, \dot{h}(t))\, dt \\ & + \tfrac{1}{2}((\Delta\, \vec{\nabla}F_t - \vec{\nabla}\, \Delta F_t)(\Sigma(t)), U(t)\, h(t))\, dt. \end{aligned}$$

This equation and the Bochner Wietzenbock Formula in Eq. (2.6) proves the proposition. Q.E.D.

Notation 4.8. For each unitary map $U: H_0(\mathfrak{g}) \to H_0(\mathfrak{g})$, let $\mathrm{Ric}_U \equiv U^{-1}\,\mathrm{Ric}\,U$, where Ric is defined in Eq. (2.2).

COROLLARY 4.9. *Continuing the notation from Proposition* 4.7,

$$E(\vec{\nabla} f(\Sigma(T)),\, U(T)\, h(T)) = E\int_0^T (\vec{\nabla} F_t(\Sigma(t)),\, U(t)\, \dot{H}(t))\, dt,$$

where

$$H(t) \equiv h(t) + \tfrac{1}{2}\int_0^t \mathrm{Ric}_{U(\tau)}\, h(\tau)\, d\tau. \tag{4.16}$$

Proof. By Proposition 4.7 and the assumption that $h(0)=0$,

$$\begin{aligned}(\vec{\nabla} f(\Sigma(T)),\, U(T)\, h(T)) &= \int_0^T ((\nabla_{d\beta(t)} \vec{\nabla} F_t)(\Sigma(t)),\, U(t)\, h(t)) \\ &\quad + \int_0^T (\vec{\nabla} F_t(\Sigma(t)),\, U(t)\, \dot{H}(t))\, dt.\end{aligned}$$

The proof is completed by taking expectations of both sides of this equation. Q.E.D.

We now may state the first version of the main theorem of this section.

THEOREM 4.10 (Integration by Parts I). *For each* $h \in H(H_0(\mathfrak{g}))$ *and* $f \in \mathscr{F}C^\infty(\mathscr{L})$,

$$E[(\vec{\nabla} f(\Sigma(T)),\, U(T)\, h(T))] = E[f(\Sigma(T))\, z_T(h)],$$

where

$$\begin{aligned} z_T(h) &\equiv \int_0^T (U(t)\,(\dot{h}(t) + \tfrac{1}{2}\mathrm{Ric}_{U(t)}\, h(t)),\, d\beta(t)) \\ &= \int_0^T (U(t)\,\dot{h}(t) + \tfrac{1}{2}\mathrm{Ric}\; U(t)\, h(t),\, d\beta(t)). \end{aligned} \tag{4.17}$$

Proof. Let H be defined as in Eq. (4.16). Then using the L^2-isometry property of the Itô integral and Itô's lemma (Theorem 3.10) we find

$$\begin{aligned} &E[f(\Sigma(T))\, z_T(h)] \\ &= E\left[\left\{F_0(\Sigma(0)) + \int_0^T (\vec{\nabla} F_t(\Sigma(t)), d\beta(t))\right\} \int_0^T (U(t)\, \dot{H}(t), d\beta(t))\right] \\ &= E\left[\int_0^T (\vec{\nabla} F_t(\Sigma(t)), U(t)\, \dot{H}(t))\, dt\right] \\ &= E(\vec{\nabla} f(\Sigma(T)), U(T)\, h(T)), \end{aligned}$$

wherein the last equality we have used Corollary 4.9. Q.E.D.

Definition 4.11. *For each $h \in H(H_0(\mathfrak{g}))$ let X^h denote the vector-filed on $W(\mathscr{L}(G))$ defined by*

$$X_t^h(\Sigma) = L_{\Sigma(t)*}\, U(t)\, h(t).$$

Theorem 4.12 (Integration by Parts II). *For each $h \in H(H_0(\mathfrak{g}))$ and $f \in \mathscr{F} C^\infty(W(\mathscr{L}))$,*

$$E[(X^h f)(\Sigma)] = E[f(\Sigma(T))\, z_T(h)], \tag{4.18}$$

where $X^h f$ is defined in Eq. (1.12) *and $z_T(h)$ is defined in Eq.* (4.17).

Proof. Write $f(\Sigma) = F(\vec{\Sigma})$ where $\vec{\Sigma} \equiv (\Sigma(t_1), ..., \Sigma(t_k))$, $0 < t_1 < t_2 < \cdots < t_k$, and $F: \mathscr{L}^k \to \mathbb{R}$ is a smooth cylinder function. That is

$$F(g_1, ..., g_k) = \tilde{F}((g_1)_{\mathscr{P}}, ..., (g_k)_{\mathscr{P}}) \qquad \forall g_i \in \mathscr{L},$$

where $\mathscr{P} = \{0 < s_1 < s_2 < \cdots < s_n < 1\}$ is partition of $[0, 1]$ and $\tilde{F} \in C^\infty(G^{kn})$. We will prove the theorem by induction on k. The case $k = 1$ is the content of Theorem 4.10. Suppose $k > 1$ and the theorem is true when there are $k-1$ t_i's. The induction step will be completed by showing that Eq. (4.18) holds for $f(\Sigma) = F(\vec{\Sigma})$ described above.

For $h \in H_0(\mathfrak{g})$, let $\tilde{h}^{(i)} F$ denote the action of $\tilde{h}$ on the ith variable of F, i.e.,

$$(\tilde{h}^{(i)} F)(\vec{\Sigma}) \equiv \frac{d}{d\varepsilon}\bigg|_0 F((\Sigma(t_1), ..., \Sigma(t_i)\, e^{\varepsilon h}, ..., \Sigma(t_k)) \qquad \forall h \in H_0(\mathfrak{g}).$$

Also let

$$\Delta^{(i)}F \equiv \sum_{h \in S_0} (\tilde{h}^{(i)})^2 F$$

and $(\vec{\nabla}^{(i)}F)(\vec{\Sigma})$ denote the gradient of F in the ith variable, i.e., $(\vec{\nabla}^{(i)}F)(\vec{\Sigma})$ is the unique element of $H_0(\mathfrak{g})$ such that

$$((\vec{\nabla}^{(i)}F)(\vec{\Sigma}), h) = (\tilde{h}^{(i)}F)(\vec{\Sigma}) \qquad \forall h \in H_0(\mathfrak{g}).$$

Then

$$\begin{aligned} E(X^h f)(\Sigma) &= \sum_{i=1}^{k-1} E((\vec{\nabla}^{(i)}F)(\vec{\Sigma}), U(t_i)\, h(t_i)) + E((\vec{\nabla}^{(k)}F)(\vec{\Sigma}), U(t_k)\, h(t_k)) \\ &=: S + T. \end{aligned} \tag{4.19}$$

Let $\sigma \equiv (\Sigma(t_1), ..., \Sigma(t_{k-1}))$, $u = t_{k-1}$, $v = t_k$, $\delta \equiv t_k - t_{k-1}$, and $F_t(\sigma, g) \equiv (e^{(v-t)\,\Delta^{(k)}/2}F)(\sigma, g)$. Then by Proposition 4.7 and Corollary 4.9,

$$\begin{aligned} T &= E\left[((\vec{\nabla}^{(k)}F_u)(\sigma, \Sigma(u)), U(u)\, h(u)) + \int_u^v d((\vec{\nabla}^{(k)}F_t)(\sigma, \Sigma(t)), U(t)\, h(t))\right] \\ &= E\left[((\vec{\nabla}^{(k)}F_u)(\sigma, \Sigma(u)), U(u)\, h(u)) + \int_u^v ((\vec{\nabla}^{(k)}F_t)(\sigma, \Sigma(t)), U(t)\, \dot{H}(t))\, dt\right] \\ &= E\left[((\vec{\nabla}^{(k)}F_u)(\sigma, \Sigma(u)), U(u)\, h(u))\right. \\ &\quad \left. + \int_u^v ((\vec{\nabla}^{(k)}F_t)(\sigma, \Sigma(t)), d\beta(t)) \cdot \int_u^v (U(t)\, \dot{H}(t), d\beta(t))\right]. \end{aligned}$$

By Theorem 3.14,

$$\begin{aligned} \int_u^v ((\vec{\nabla}^{(k)}F_t)(\sigma, \Sigma(t)), d\beta(t)) &= F_v(\sigma, \Sigma(v)) - F_u(\sigma, \Sigma(u)) \\ &= F(\vec{\Sigma}) - F_u(\sigma, \Sigma(u)). \end{aligned}$$

Since $v \to \int_0^v (U(t)\, \dot{H}(t), d\beta(t))$ is a Martingale,

$$E[F_u(\sigma, \Sigma(u)) \int_u^v (U(t)\, \dot{H}(t), d\beta(t))] = 0.$$

Combining the three above displayed equations gives

$$T = E\left\{((\vec{\nabla}^{(k)} F_u)(\sigma, \Sigma(u)), U(u)\, h(u)) + F(\vec{\Sigma}) \cdot \int_u^v (U(t)\, \dot{H}(t), d\beta(t))\right\}. \quad (4.20)$$

Using the Markov property, S may be written as

$$S = \sum_{i=1}^{k-1} E((\vec{\nabla}^{(i)} F_u)(\sigma, \Sigma(u)), U(t_i)\, h(t_i)). \quad (4.21)$$

Set $V(\Sigma) = F_u(\sigma, \Sigma(u))$ so that $V \in \mathcal{F}C^\infty(W(\mathcal{L}))$, and notice that

$$(X^h V)(\Sigma) = \sum_{i=1}^{k-1} ((\vec{\nabla}^{(i)} F_u)(\sigma, \Sigma(u)), U(t_i)\, h(t_i)) + ((\vec{\nabla}^{(k)} F_u)(\sigma, \Sigma(u)), U(u)\, h(u)). \quad (4.22)$$

Therefore, by Equations (4.20–4.22),

$$\begin{aligned} S + T &= E\left\{(X^h V)(\Sigma) + F(\vec{\Sigma}) \cdot \int_u^v (U(t)\, \dot{H}(t), d\beta(t))\right\} \\ &= E\left\{V(\Sigma) \int_0^u (U(t)\, \dot{H}(t), d\beta(t)) + F(\vec{\Sigma}) \cdot \int_u^v (U(t)\, \dot{H}(t), d\beta(t))\right\}, \end{aligned}$$

wherein the second equality we have used the induction hypothesis. Using the Markov property once again,

$$\begin{aligned} E\left\{V(\Sigma) \int_0^u (U(t)\, \dot{H}(t), d\beta(t))\right\} &= E\left\{F_u(\sigma, \Sigma(u)) \int_0^u (U(t)\, \dot{H}(t), d\beta(t))\right\} \\ &= E\left\{F(\sigma, \Sigma(v)) \int_0^u (U(t)\, \dot{H}(t), d\beta(t))\right\} \\ &= E\left\{F(\vec{\Sigma}) \int_0^u (U(t)\, \dot{H}(t), d\beta(t))\right\}. \end{aligned}$$

The Theorem now follows from the last two equations and (4.19). Q.E.D.

4.3. *Closability of the Dirichlet Form*

Recall that ν_T is the Law of $\Sigma_T = \Sigma(T)$, where Σ is the Brownian motion on $\mathcal{L}$ constructed in Theorem 3.8 with $\Sigma(0, s) \equiv e$ for all $s \in [0, 1]$.

DEFINITION 4.13. Let $\mathscr{E}^0_T$ denote the symmetric quadratic form on $L^2(\mathscr{L}(G), \nu_T)$ with domain $\mathscr{D}(\mathscr{E}^0_T) = \mathscr{F}C^\infty(\mathscr{L})$ and for $u, v \in \mathscr{F}C^\infty(\mathscr{L})$,

$$\mathscr{E}^0_T(u, v) \equiv \int_{\mathscr{L}(G)} (\vec{\nabla} u(g), \vec{\nabla} v(g))_{H_0(\mathfrak{g})} \, \nu_T(dg) = E[(\vec{\nabla} u(\Sigma_T), \vec{\nabla} v(\Sigma_T))_{H_0(\mathfrak{g})}].$$

THEOREM 4.14. *The quadratic form $\mathscr{E}^0_T$ is closable.*

Proof. To simplify notation, let $\mathscr{E}^0_T(f) = \mathscr{E}^0_T(f, f)$. Suppose that $f_n \in \mathscr{F}C^\infty(\mathscr{L})$ such that $\lim_{n \to \infty} f_n = 0$ in $L^2(\nu_T)$ and

$$\mathscr{E}^0_T(f_n - f_n) \to 0 \qquad \text{as} \quad m, n \to \infty. \tag{4.23}$$

We must show that $\lim_{n \to \infty} \mathscr{E}^0_T(f_n) = 0$. Because of (4.23) the functions $G_n \equiv \vec{\nabla} f_n \in \mathscr{F}C^\infty(\mathscr{L}, H_0(\mathfrak{g}))$ form a Cauchy sequence in $L^2(\mathscr{L}(G), \nu_T; H_0(\mathfrak{g}))$. Hence there exists $G \in L^2(\mathscr{L}(G), \nu_T; H_0(\mathfrak{g}))$ such that $L^2-\lim_{n \to \infty} G_n = G$. Since

$$\mathscr{E}^0_T(f_n) = \int_{\mathscr{L}(G)} \|G_n\|^2 \, d\nu_T = E \, \|G_n(\Sigma_T)\|^2,$$

it follows that $\lim_{n \to \infty} \mathscr{E}^0_T(f_n) = E \, \|G(\Sigma_T)\|^2$. So to finish the proof it suffices to show that $G(\Sigma_T) = 0$ a.s.

To this end let $h \in H_0(\mathfrak{g})$, $Q \in \mathscr{F}C^\infty(W(\mathscr{L}))$, and set $k(t) = (t/T)\, h$. Then $k \in H(H_0(\mathfrak{g}))$ and using the integration by parts Theorem 4.10 we find

$$\begin{aligned} E\{(G(\Sigma(T)), U(T)\, h)\, Q(\Sigma)\} &= \lim_{n \to \infty} E\{(\vec{\nabla} f_n(\Sigma(T)), U(T)\, k(T))\, Q(\Sigma)\} \\ &= \lim_{n \to \infty} E\{(X^k f_n(\Sigma_T))\, Q(\Sigma)\} \\ &= \lim_{n \to \infty} E\{f_n(\Sigma_T)(-X^k + z_T(k))\, Q(\Sigma)\} = 0. \end{aligned}$$

Because $\mathscr{F}C^\infty(W(\mathscr{L}))$ is dense in $L^2(W(\mathscr{L}), \nu)$ and because $Q \in \mathscr{F}C^\infty(W(\mathscr{L}))$ was arbitrary, the last displayed equation implies $(G(\Sigma_T), U(T)\, h) = 0$ a.s. Hence

$$\|G(\Sigma_T)\|^2 = \sum_{h \in S_0} (G(\Sigma_T), U(T)\, h)^2 = 0 \qquad \text{a.s.},$$

i.e., $G(\Sigma_T) \equiv 0$ a.s. Q.E.D.

Remark 4.15. Theorem 4.14 may be stated equivalently as saying that the gradient operator $\vec{\nabla}$ with domain $\mathscr{F}C^\infty(\mathscr{L})$ has a densely defined L^2-adjoint. However, the method of proof does not give any explicit information as to what is in the domain in $L^2(\mathscr{L}, \nu_T; H_0(\mathfrak{g}))$ of the adjoint operator $\vec{\nabla}^*$. This deficiency will be remedied in Theorem 6.2 of Section 6 below where it is shown that $\mathscr{F}C^\infty(\mathscr{L}, H_0(\mathfrak{g})) \subset \mathscr{D}(\vec{\nabla}^*)$.

5. THE FINITE DIMENSIONAL APPROXIMATIONS

5.1. *Finite Dimensional Integration by Parts Formula*

In this section let $\mathscr{P} = \{0 < s_1 < s_2 < \cdots < s_n < 1\}$ be a partition of $[0, 1]$. In order to prove Theorem 1.5 above, we will apply Corollary 6.5 of Driver [5] to the Lie group $G^{\mathscr{P}}$ and then pass to the limit of finer and finer partitions $\mathscr{P}$. In order to carry out this procedure it is necessary to introduce the unique Riemannian metric, $(\cdot, \cdot)_{\mathscr{P}}$, on $G^{\mathscr{P}}$ for which $\Delta_{\mathscr{P}}$ in Eq. (3.15) will become the Laplace Beltrami operator on $(G^{\mathscr{P}}, (\cdot, \cdot)_{\mathscr{P}})$.

Let $\mathfrak{g}^{\mathscr{P}}$ be the Lie algebra of $G^{\mathscr{P}}$ which may naturally be identified with the set of functions from $\mathscr{P}$ to $\mathfrak{g}$. In the sequel, h and k will typically denote elements of $\mathfrak{g}^{\mathscr{P}}$ or $H_0(\mathfrak{g})$.

PROPOSITION 5.1 (Metric on $G^{\mathscr{P}}$). *Let $\mathscr{P} = \{0 < s_1 < s_2 < \cdots < s_n < 1\}$ be a partition of $[0, 1]$, and Q be the inverse to the matrix $\{G_0(s_i, s_j)\}_{i,j=1}^n$, and*

$$(h, k)_{\mathscr{P}} \equiv \sum_{i,j=1}^{n} Q_{ij} \langle h(s_i), k(s_j) \rangle \qquad \text{for all} \quad h, k \in \mathfrak{g}^{\mathscr{P}}.$$

We extend $(\cdot, \cdot)_{\mathscr{P}}$ to a left invariant Riemannian metric on $G^{\mathscr{P}}$ which will still be denoted by $(\cdot, \cdot)_{\mathscr{P}}$. Then the elliptic differential operator $\Delta_{\mathscr{P}}$ defined in (3.15) is the Laplace Beltrami operator on $G^{\mathscr{P}}$ with metric $(\cdot, \cdot)_{\mathscr{P}}$.

Proof. It is an exercise in linear algebra to check that $\Delta_{\mathscr{P}}$ may be written as $\Delta_{\mathscr{P}} = \sum_{h \in \Gamma} \tilde{h}^2$, where Γ is an orthonormal basis of $(\mathfrak{g}^{\mathscr{P}}, (\cdot, \cdot)_{\mathscr{P}})$ and $\tilde{h}$ denotes the unique left invariant vector field on $G^{\mathscr{P}}$ such that $\tilde{h}(e) = h$. It is well known that $\sum_{h \in \Gamma} \tilde{h}^2$ is the Laplace-Beltrami operator on $G^{\mathscr{P}}$ because $G^{\mathscr{P}}$ is compact and hence uni-modular, see for example Remark 2.2 in Driver and Gross [7]. Q.E.D.

Notation 5.2. Let $\beta_{\mathscr{P}}(t)$ be the standard Brownian motion on $(\mathfrak{g}^{\mathscr{P}}, (\cdot, \cdot)_{\mathscr{P}})$ given by $\beta_{\mathscr{P}}(t) = (\beta(t, s_1), \beta(t, s_2), ..., \beta(t, s_n))$, and $\nabla^{\mathscr{P}}$ and $\mathrm{Ric}_{\mathscr{P}}$ be the Levi-Civita covariant derivative and the Ricci tensor on $(G^{\mathscr{P}}, (\cdot, \cdot)_{\mathscr{P}})$ respectively. For all $h, k \in \mathfrak{g}^{\mathscr{P}}$, define $D_h^{\mathscr{P}} k = (\nabla_{\tilde{h}}^{\mathscr{P}} \tilde{k})(e)$ with $e \in G^{\mathscr{P}}$ being the identity element. Also define $U_{\mathscr{P}}$ to the solution to the Stratonovich differential equation,

$$dU_{\mathscr{P}}(t) + D_{\delta\beta_{\mathscr{P}}(t)}^{\mathscr{P}} U_{\mathscr{P}}(t) = 0 \qquad \text{with} \quad U_{\mathscr{P}}(0) = I \in End(\mathfrak{g}_{\mathscr{P}}).$$

The operators $D_h^{\mathscr{P}}$ and $\mathrm{Ric}_{\mathscr{P}}$ are computed explicitly in Proposition 5.7 and Eq. (5.9) below. We now may state a finite dimensional verion of Theorem 1.5. Q.E.D.

THEOREM 5.3. *Let $T>0$ and $l: [0, T] \to \mathbb{R}$ be an absolutely continuous function such that $l(0)=0$, $l(T)=1$ and $\int_0^T \dot{l}^2(t)\,dt < \infty$. Suppose that $f \in \mathscr{F}C^\infty(\mathscr{L})$ is of the form $f(g) = F(g_{\mathscr{P}})$ where $F \in C^\infty(G^{\mathscr{P}})$ and $\mathscr{P} = \{0 < s_1 < s_2 < \cdots < s_n < 1\}$ is a partition of $[0, 1]$. Then for all $h \in H_0(\mathfrak{g})$,*

$$E[(\tilde{h}f)(\Sigma(T))] = E\left[f(\Sigma(T))(U_{\mathscr{P}}(T)^{-1} h_{\mathscr{P}}, \int_0^T U_{\mathscr{P}}^{-1}(t)(\dot{l}(t) - \tfrac{1}{2} l(t)\,\mathrm{Ric}_{\mathscr{P}})\, d\overleftarrow{\beta}_{\mathscr{P}}(t))\right], \quad (5.1)$$

where $h_{\mathscr{P}} \equiv h|_{\mathscr{P}}$ and $d\overleftarrow{\beta}_{\mathscr{P}}$ denotes the backwards Itô differential.

Proof. By Proposition 5.1 and the dicussion leading up to Eq. (3.15), $\Sigma_{\mathscr{P}}$ and $\beta_{\mathscr{P}}$ are Brownian motions on $G^{\mathscr{P}}$ and $\mathfrak{g}^{\mathscr{P}}$ respectively which are related to each other by Eq. 3.14. Using this remark and the identities, $f(\Sigma(T)) = F(\Sigma_{\mathscr{P}}(t))$ and $(\tilde{h}f)(\Sigma(T)) = (\tilde{h}_{\mathscr{P}}F)(\Sigma_{\mathscr{P}}(t))$, Eq. (5.1) follows as an application of Corollary 6.5 in [5]. Q.E.D.

The proof of Theorem 1.5 will be given in Section 6 by passing to the limit of finer and finer partitions $\mathscr{P}$ of $[0, 1]$ in Eq. (5.1). In order to take this limit it is necessary to understand the geometry on $G^{\mathscr{P}}$ and its relationship to the geometry on $\mathscr{L}(G)$. This is the topic of the next subsection.

5.2. *Geometry of the Finite Dimensional Approximations*

To facilitate our computations, it will be convenient to identify $\mathfrak{g}^{\mathscr{P}}$ with the orthogonal compliment to the null space, $\mathrm{nul}(\Lambda_{\mathscr{P}}) \subset H_0(\mathfrak{g})$, where $\Lambda_{\mathscr{P}}: H_0(\mathfrak{g}) \to \mathfrak{g}^{\mathscr{P}}$ is defined by $\Lambda_{\mathscr{P}}(h) \equiv h|_{\mathscr{P}}$. The next lemma shows that

$$\mathrm{nul}(\Lambda_{\mathscr{P}})^\perp = H_{\mathscr{P}}(\mathfrak{g}) \equiv \{h \in H_0(\mathfrak{g}) \cap C^2((0, 1)\backslash\mathscr{P}, \mathfrak{g}): h'' = 0 \text{ on } [0, 1]\backslash\mathscr{P}\}.$$

(Notice that $h \in H_{\mathscr{P}}(\mathfrak{g})$ iff $h \in H_0(\mathfrak{g})$ and h is piecewise linear.) The following notation will be used in the next lemma and the remainder of this section.

Notation 5.4. Given a partition $\mathscr{P} = \{0 < s_1 < s_2 < \cdots < s_n < 1\}$ of $[0, 1]$ and $h \in H_0(\mathfrak{g})$, for $i = 0, 1, 2, ..., n$, let $\delta_i h \equiv h(s_{i+1}) - h(s_i)$ and $\delta_i \equiv s_{i+1} - s_i$ where $s_0 \equiv 0$ and $s_{n+1} \equiv 1$.

LEMMA 5.5. *The orthogonal compliment $H_{\mathscr{P}}(\mathfrak{g})^\perp$ of $H_{\mathscr{P}}(\mathfrak{g})$ in $H_0(\mathfrak{g})$ is*

$$H_{\mathscr{P}}(\mathfrak{g})^\perp = \mathrm{nul}(\Lambda_{\mathscr{P}}) = \{h \in H_0(\mathfrak{g}): h|_{\mathscr{P}} \equiv 0\}.$$

Proof. Suppose that $h \in H_0(\mathfrak{g})$ and $k \in H_{\mathscr{P}}(\mathfrak{g})$, then

$$(h, k) = \sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} \langle h'(\sigma), k'(\sigma) \rangle \, d\sigma = \sum_{i=0}^{n} \langle \delta_i h, \delta_i k \rangle / \delta_i. \tag{5.2}$$

Hence if $h \in \operatorname{nul}(\Lambda_{\mathscr{P}})$, i.e., $h|_{\mathscr{P}} = 0$, then $(h, k) = 0$ for all $k \in H_{\mathscr{P}}(\mathfrak{g})$. Hence $\operatorname{nul}(\Lambda_{\mathscr{P}}) \subset H_{\mathscr{P}}(\mathfrak{g})^{\perp}$.

For the other inclusion, suppose that $A_i \in \mathfrak{g}$ is given such that $\sum_{i=0}^{n} A_i = 0$. Define

$$k(s) \equiv \int_0^s \left(\sum_{i=0}^{n} 1_{(s_i, s_{i+1}]}(\sigma) \, A_i/\delta_i \right) d\sigma = \sum_{i=0}^{n} (s_{i+1} \wedge s - s_i \wedge s) \, A_i/\delta_i.$$

Since $k'(s) = \delta_i k/\delta_i = A_i/\delta_i$ for $s \in (s_i, s_{i+1}]$ and $k(1) = \sum_{i=0}^{n} A_i = 0$, k is in $H_{\mathscr{P}}(\mathfrak{g})$. If $h \in H_{\mathscr{P}}(\mathfrak{g})^{\perp}$ then, using the k just constructed in Eq. (5.2),

$$0 = \sum_{i=0}^{n} \langle \delta_i h, A_i \rangle / \delta_i \tag{5.3}$$

for all $A_i \in \mathfrak{g}$ such that $\sum_{i=0}^{n} A_i = 0$. Since

$$\sum_{i=0}^{n} \delta_i h = h(1) - h(0) = 0,$$

we may put $A_i \equiv \delta_i h$ in (5.3) to find

$$0 = \sum_{i=0}^{n} \langle \delta_i h, \delta_i h \rangle / \delta_i,$$

i.e., $\delta_i h \equiv 0$ for all $i = 0, 1, 2, ..., n-1$. Because $h(0) = 0$, this implies that $h|_{\mathscr{P}} \equiv 0$. Thus we have shown that if $H_{\mathscr{P}}(\mathfrak{g})^{\perp} \subset \operatorname{nul}(\Lambda_{\mathscr{P}})$. Q.E.D.

In general $H_{\mathscr{P}}(\mathfrak{g})$ is a subspace of $H_0(\mathfrak{g})$ but not a Lie subalgebra with the inherited pointwise commutator. In order to remedy this, let $P_{\mathscr{P}}$: $H_0(\mathfrak{g}) \to H_{\mathscr{P}}(\mathfrak{g})$ denote the orthogonal projection map and define $[\cdot, \cdot]_{\mathscr{P}}$ on $H_{\mathscr{P}}(\mathfrak{g})$ by $[h, k]_{\mathscr{P}} \equiv P_{\mathscr{P}}[h, k]$. One may check that $(H_{\mathscr{P}}(\mathfrak{g}), [\cdot, \cdot]_{\mathscr{P}})$ is a Lie algebra. Indeed the only non-trivial property to verify is the Jacobi identity. Since $[h, [k, l]_{\mathscr{P}}]_{\mathscr{P}} = P_{\mathscr{P}}[h, [k, l]_{\mathscr{P}}]$ is uniquely determined by it values on $\mathscr{P}$, i.e., by the values $[h(s), [k, l]_{\mathscr{P}}(s)] = [h(s), [k(s), l(s)]]$ for $s \in \mathscr{P}$, the Jacobi identity simply follows from the Jacobi identity for the Lie bracket $([\cdot, \cdot])$ on $\mathfrak{g}$.

LEMMA 5.6. *Let $H_{\mathscr{P}}(\mathfrak{g})$ and $\mathfrak{g}^{\mathscr{P}}$ be the Lie algebras described above equipped with the inner products $(\cdot, \cdot) = (\cdot, \cdot)_{H_0(\mathfrak{g})}$ and $(\cdot, \cdot)_{\mathscr{P}}$ respectively.*

Then linear map $\Lambda_{\mathcal{P}}: H_{\mathcal{P}}(\mathfrak{g}) \to \mathfrak{g}^{\mathcal{P}}$ $(\Lambda_{\mathcal{P}}(h) \equiv h_{\mathcal{P}} \equiv h|_{\mathcal{P}})$ *is an isometric Lie algebra isomorphism.*

Proof. The only assertion which is not obvious to check the assertion that $\Lambda_{\mathcal{P}}$ is an isometry. For $A = (A_1, A_2, ..., A_n) \in \mathfrak{g}^n$ let $h_A \equiv \sum_{i=1}^n G_0(s_i, \cdot)\, A_i$ and notice that $h_A \in H_{\mathcal{P}}(\mathfrak{g})$. Using the reproducing kernel property for G_0, see Eq. 3.11 in [6],

$$(h_A, h_B) = \sum_{i,j=1}^{n} (G_0(s_i, \cdot), G_0(s_j, \cdot))_{H_0(\mathbb{R})} \langle A_i, B_j \rangle = \sum_{i,j=1}^{n} G_0(s_i, s_j) \langle A_i, B_j \rangle.$$

Because $\{G_0(s_i, s_j)\}_{i,j=1}^n$ is a positive definite matrix, the last equation with $B = A$, shows that $A \in \mathfrak{g}^n \to h_A \in H_{\mathcal{P}}(\mathfrak{g})$ is injective hence surjective by the rank nullity theorem. On the other hand,

$$\begin{aligned}(\Lambda_{\mathcal{P}} h_A, \Lambda_{\mathcal{P}} h_B)_{\mathcal{P}} &= \sum_{i,j,k,l=1}^{n} Q_{k,l} \langle h_A(s_k), h_B(s_l) \rangle \\ &= \sum_{i,j,k,l=1}^{n} Q_{k,l}\, G_0(s_i, s_k)\, G_0(s_j, s_l) \langle A_i, B_j \rangle \\ &= \sum_{i,j=1}^{n} G_0(s_i, s_j) \langle A_i, B_j \rangle.\end{aligned}$$

Comparing the last two displayed equations proves the isometry assertion. Q.E.D.

Alternate proof of the isometry property. In this proof we will use the fact that second order elliptic differential operators on a manifold induce a unique Riemannian metric on the manifold.

Let $F \in C^\infty(G^{\mathcal{P}})$, $f \equiv F \circ \pi_{\mathcal{P}} \in \mathcal{F}C^\infty(\mathcal{L})$, and $S_{\mathcal{P}}$ be an orthonormal basis for $H_{\mathcal{P}}(\mathfrak{g})$. Then, using Lemma 5.5 and the fact that the sum defining the Laplace operator Δ in Eq. (2.1) is basis independent, we have

$$\Delta f(g) = \sum_{h \in S_{\mathcal{P}}} (\tilde{h}^2 f)(g) = \sum_{h \in S_{\mathcal{P}}} (\tilde{h}^2_{\mathcal{P}} F)(g_{\mathcal{P}}) \qquad \forall g \in \mathcal{L}.$$

On the other hand by Eq. (2.3), $\Delta f(g) = (\Delta_{\mathcal{P}} F)(g_{\mathcal{P}})$. Hence we learn that $\Delta_{\mathcal{P}} = \sum_{h \in S_{\mathcal{P}}} \tilde{h}^2_{\mathcal{P}}$ which is the Laplace Beltrami operator on $G^{\mathcal{P}}$ equipped with the metric on $\mathfrak{g}^{\mathcal{P}}$ for which the map $\Lambda_{\mathcal{P}}$ is an isometry. But this inner product must agree with $(\cdot, \cdot)_{\mathcal{P}}$, since we have seen in the proof of Proposition 5.1 that $\Delta_{\mathcal{P}}$ is also the Laplacian relative to the metric $(\cdot, \cdot)_{\mathcal{P}}$ on $G^{\mathcal{P}}$. Q.E.D.

In the sequel we will identify $\mathfrak{g}^{\mathcal{P}}$ with $H_{\mathcal{P}}(\mathfrak{g})$.

PROPOSITION 5.7. *For $h \in H_{\mathscr{P}}(\mathfrak{g})$, let $D_h^{\mathscr{P}}: H_{\mathscr{P}}(\mathfrak{g}) \to H_{\mathscr{P}}(\mathfrak{g})$ denote Lie algebra version of the Levi-Civita covariant derivative $(\nabla^{\mathscr{P}})$ on $G^{\mathscr{P}}$ as defined in Notation 5.2 (Recall that we are identifying $\mathfrak{g}^{\mathscr{P}}$ with $H_{\mathscr{P}}(\mathfrak{g})$ as in Lemma 5.6.) Then $D_h^{\mathscr{P}} = P_{\mathscr{P}} D_h$, where as above $P_{\mathscr{P}}$ is the orthogonal projection of $H_0(\mathfrak{g})$ onto $H_{\mathscr{P}}(\mathfrak{g})$.*

Proof. We need to check that $D_h^{\mathscr{P}}$ is metric compatible and Torsion free. Both of these properties follow directly from the corresponding properties of D_h described in Theorem 3.12 of Driver and Lohrenz [6]. Indeed if $h, k \in H_{\mathscr{P}}(\mathfrak{g})$, then

$$(D_h^{\mathscr{P}} k, k) = (P_{\mathscr{P}} D_h k, k) = (D_h k, k) = 0$$

and

$$D_h^{\mathscr{P}} k - D_k^{\mathscr{P}} h = P_{\mathscr{P}}(D_h k - D_k h) = P_{\mathscr{P}}([h, k]) = [h, k]_{\mathscr{P}}. \qquad \text{Q.E.D.}$$

THEOREM 5.8. *Let $S_{\mathscr{P}}$ be an orthonormal basis for $H_{\mathscr{P}}(\mathfrak{g})$ and $\Delta_{\mathscr{P}}^{(1)}: H_{\mathscr{P}}(\mathfrak{g}) \to H_{\mathscr{P}}(\mathfrak{g})$ be defined by*

$$\Delta_{\mathscr{P}}^{(1)} \equiv \sum_{k \in S_{\mathscr{P}}} D_k^{\mathscr{P}} D_k^{\mathscr{P}}.$$

Then

$$\lim_{|\mathscr{P}| \to 0} \| P_{\mathscr{P}}(\Delta^{(1)} - \Delta_{\mathscr{P}}^{(1)}) P_{\mathscr{P}} \|_{op} = 0,$$

where $\|\cdot\|_{op}$ is the operator norm on bounded linear operators on $H_0(\mathfrak{g})$.

The following lemma is used in the proof of this theorem.

LEMMA 5.9. *Let $S_{\mathscr{P}}$ be an orthonormal basis for $H_{\mathscr{P}}(\mathfrak{g})$ and $A, B \in \mathfrak{g}$. Then*

$$\sum_{k \in S_{\mathscr{P}}} \langle [k(s_i), A], [k(s_j), B] \rangle = G_0(s_i, s_j) K\langle A, B \rangle \tag{5.4}$$

for all $i, j \in \{1, 2, ..., n\}$.

Proof. It is easily checked that the left member in Eq. (5.4) is independent of the orthonormal basis $S_{\mathscr{P}}$ of $H_{\mathscr{P}}(\mathfrak{g})$. So to simplify the computation, we may take $S_{\mathscr{P}} \equiv \{lC\}_{l \in \mathfrak{h}_{\mathscr{P}},\, C \in \mathfrak{g}_0}$ where $\mathfrak{g}_0$ is an orthonormal basis for $\mathfrak{g}$ and $\mathfrak{h}_{\mathscr{P}}$ is an orthonormal basis for

$$H_{\mathscr{P}}(\mathbb{R}) \equiv \{l \in H_0(\mathbb{R}) \cap C^2((0,1)\backslash\mathscr{P}, \mathbb{R}) : l'' \equiv 0 \text{ on } (0,1)\backslash\mathscr{P}\}.$$

Because

$$\begin{aligned}\sum_{C \in \mathfrak{g}_0} \langle [C, A], [C, B] \rangle &= \sum_{C \in \mathfrak{g}_0} \langle ad_A C, ad_B C \rangle = \sum_{C \in \mathfrak{g}_0} -\langle C, ad_A ad_B C \rangle \\ &= -\operatorname{tr}(ad_A ad_B) = K\langle A, B \rangle,\end{aligned}$$

it follows that

$$\begin{aligned}\sum_{k \in S_{\mathscr{P}}} \langle [k(s_i), A], [k(s_j), B] \rangle &= \sum_{l \in \mathfrak{h}_{\mathscr{P}},\, C \in \mathfrak{g}_0} l(s_i)\, l(s_j) \langle [C, A], [C, B] \rangle \\ &= \sum_{l \in \mathfrak{h}_{\mathscr{P}}} l(s_i)\, l(s_j)\, K\langle A, B \rangle.\end{aligned}$$

To evaluate $\sum_{l \in \mathfrak{h}_{\mathscr{P}}} l(s_i)\, l(s_j)$, let $\mathfrak{h} \subset H_0(\mathbb{R})$ be an orthonormal basis of $H_0(\mathbb{R})$ which contains $\mathfrak{h}_{\mathscr{P}}$. Notice if $l \in \mathfrak{h}\backslash\mathfrak{h}_{\mathscr{P}}$, then $l \in H_{\mathscr{P}}(\mathbb{R})^{\perp}$ and hence $l|_{\mathscr{P}} \equiv 0$. Therefore

$$\sum_{l \in \mathfrak{h}_{\mathscr{P}}} l(s_i)\, l(s_j) = \sum_{l \in \mathfrak{h}} l(s_i)\, l(s_j) = G_0(s_i, s_j),$$

where the last equality verified in Lemma 3.8 of [6]. Q.E.D.

Proof of Theorem 5.8. Let $h, J \in H_{\mathscr{P}}(\mathfrak{g})$, then

$$\begin{aligned}(\Delta_{\mathscr{P}}^{(1)} h, J) &= \sum_{k \in S_{\mathscr{P}}} (D_k^{\mathscr{P}} D_k^{\mathscr{P}} h, J) \\ &= -\sum_{k \in S_{\mathscr{P}}} (D_k^{\mathscr{P}} h, D_k^{\mathscr{P}} J) \\ &= -\sum_{k \in S_{\mathscr{P}}} \sum_{i=0}^{n} \langle \delta_i(D_k h), \delta_i(D_k J) \rangle / \delta_i.\end{aligned}$$

Now

$$\delta_i(D_k h) = \int_{s_i}^{s_{i+1}} [k, dh] - \delta_i \int_0^1 [k, dh]$$

and

$$\begin{aligned}\int_{s_i}^{s_{i+1}} [k, dh] &= \int_{s_i}^{s_{i+1}} \left[k(s_i) + \frac{(\sigma - s_i)}{\delta_i}\, \delta_i k, \delta_i h/\delta_i \right] d\sigma \\ &= \left[k(s_i)\, \delta_i + \frac{\delta_i^2}{2\delta_i}\, \delta_i k, \delta_i h/\delta_i \right] \\ &= \left[k(s_i) + \frac{1}{2}\, \delta_i k, \delta_i h \right] \\ &= [(k(s_i) + k(s_{i+1}))/2, \delta_i h].\end{aligned}$$

Set $k_i^a \equiv (k(s_i) + k(s_{i+1}))/2$, then the above two displayed equations show that

$$\delta_i(D_k h) = [k_i^a, \delta_i h] - \delta_i \sum_{j=0}^{n} [k_j^a, \delta_j h]. \tag{5.5}$$

Thus, using Lemma 5.9,

$$\begin{aligned}(\Delta_{\mathscr{P}}^{(1)} h, J) &= - \sum_{k \in S_{\mathscr{P}}} \sum_{i=0}^{n} \delta_i^{-1} \Big\langle [k_i^a, \delta_i h] - \delta_i \sum_{j=0}^{n} [k_j^a, \delta_j h], \\ &\qquad [k_i^a, \delta_i J] - \delta_i \sum_{j=0}^{n} [k_j^a, \delta_j J] \Big\rangle \\ &= - \sum_{k \in S_{\mathscr{P}}} \sum_{i=0}^{n} \delta_i^{-1} \Big\{ \langle [k_i^a, \delta_i h], [k_i^a, \delta_i J] \rangle \\ &\qquad + \delta_i^2 \Big\langle \sum_{j=0}^{n} [k_j^a, \delta_j h], \sum_{l=0}^{n} [k_l^a, \delta_l J] \Big\rangle \Big\} \\ &\qquad - \sum_{k \in S_{\mathscr{P}}} \sum_{i=0}^{n} \delta_i^{-1} \Big\{ \delta_i \Big\langle \sum_{j=0}^{n} [k_j^a, \delta_j h], [k_i^a, \delta_i J] \Big\rangle \\ &\qquad - \delta_i \Big\langle [k_i^a, \delta_i h], \sum_{j=0}^{n} [k_j^a, \delta_j J] \Big\rangle \Big\} \\ &= - \sum_{k \in S_{\mathscr{P}}} \sum_{i=0}^{n} \{ \delta_i^{-1} \langle [k_i^a, \delta_i h], [k_i^a, \delta_i J] \rangle \\ &\qquad - \Big\langle \sum_{j=0}^{n} [k_j^a, \delta_j h], [k_i^a, \delta_j J] \Big\rangle \Big\} \\ &= \sum_{i=0}^{n} \Big\{ -G_{ii}\, \delta_i^{-1} K \langle \delta_i h, \delta_i J \rangle + \sum_{j=0}^{n} G_{ij} K \langle \delta_j h, \delta_i J \rangle \Big\} \\ &= -S + T,\end{aligned}$$

where

$$G_{ij} = \tfrac{1}{4}\{G_0(s_i, s_j) + G_0(s_{i+1}, s_j) + G_0(s_i, s_{j+1}) + G_0(s_{i+1}, s_{j+1})\}.$$

We now work on the two terms S and T separately.

$$S = \sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} G_{ii} K\langle h'(\sigma), J'(\sigma)\rangle \, d\sigma = \int_0^1 \rho_{\mathscr{P}}(\sigma) \, K\langle h'(\sigma), J'(\sigma)\rangle \, d\sigma,$$

where

$$\rho_{\mathscr{P}}(s) \equiv \sum_{i=0}^{n} G_{ii} 1_{(s_i, s_{i+1}]}(s).$$

Similarly,

$$\begin{aligned} T &= \sum_{i=0}^{n} \sum_{j=0}^{n} G_{ij} K\langle \delta_j h, \delta_i J\rangle \\ &= \sum_{i=0}^{n} \sum_{j=0}^{n} G_{ij} \int_{s_i}^{s_{i+1}} d\sigma \int_{s_j}^{s_{j+1}} ds \, K\langle h'(s), J'(s)\rangle \\ &= \int_0^1 d\sigma \int_0^1 ds \, G_{\mathscr{P}}(\sigma, s) \, K\langle h'(s), J'(\sigma)\rangle, \end{aligned}$$

where

$$G_{\mathscr{P}}(\sigma, s) \equiv \sum_{i, j=0}^{n} G_{ij} 1_{(s_i, s_{i+1}]}(\sigma) \cdot 1_{(s_j, s_{j+1}]}(s). \tag{5.6}$$

Assembling the above computations gives

$$\begin{aligned} (\varDelta_{\mathscr{P}}^{(1)} h, J) = &-\int_0^1 \rho_{\mathscr{P}}(\sigma) \, K\langle h'(\sigma), J'(\sigma)\rangle \, d\sigma \\ &+ \int_0^1 d\sigma \int_0^1 ds \, G_{\mathscr{P}}(\sigma, s) \, K\langle h'(s), J'(\sigma)\rangle. \end{aligned}$$

From Eq. (4.42) in [6],

$$\begin{aligned} (\varDelta^{(1)} h, J) = &-\int_0^1 G_0(\sigma, \sigma) \, K\langle h'(\sigma), J'(\sigma)\rangle \, d\sigma \\ &+ \int_0^1 d\sigma \int_0^1 ds \, G_0(\sigma, s) \, K\langle h'(s), J'(\sigma)\rangle. \end{aligned}$$

Combining the last two equations gives

$$((\Delta^{(1)} - \Delta^{(1)}_{\mathscr{P}})\, h, J) = \int_0^1 (\rho_{\mathscr{P}}(\sigma) - G_0(\sigma, \sigma))\, K\langle h'(\sigma), J'(\sigma)\rangle\, d\sigma$$
$$+ \int_0^1 d\sigma \int_0^1 ds (G_0(\sigma, s) - G_{\mathscr{P}}(\sigma, s))\, K\langle h'(s), J'(\sigma)\rangle.$$

It is now a simple matter to use this equation to show that

$$\|P_{\mathscr{P}}(\Delta^{(1)} - \Delta^{(1)}_{\mathscr{P}})\, P_{\mathscr{P}}\|_{B(H_{\mathscr{P}}(\mathfrak{g}))} \leqslant C\varepsilon(\mathscr{P}),$$

where

$$\varepsilon(\mathscr{P}) \equiv \max_{s, t \in [0, 1]} \{ |\rho_{\mathscr{P}}(s) - G_0(s, s)| + |G_0(s, t) - G_{\mathscr{P}}(s, t)| \}.$$

By the uniform continuity of G_0, $\lim_{|\mathscr{P}| \to 0} \varepsilon(\mathscr{P}) = 0$. This proves the theorem. Q.E.D.

We now work on the Ricci tensor.

THEOREM 5.10. *Let* $\mathrm{Ric}_{\mathscr{P}}$ *be the Lie algebra version of the Ricci tensor on* $G^{\mathscr{P}}$. (*We will interchangeably view* $\mathrm{Ric}_{\mathscr{P}}$ *as a bi-linear form or an operator on* $H_{\mathscr{P}}(\mathfrak{g})$. *Then*

$$\|P_{\mathscr{P}}(\mathrm{Ric} - \mathrm{Ric}_{\mathscr{P}})\, P_{\mathscr{P}}\|_{op} \to 0 \qquad \text{as} \quad |\mathscr{P}| \to 0.$$

Proof. Let $h \in H_{\mathscr{P}}(\mathfrak{g})$ and $S_{\mathscr{P}} \subset H_{\mathscr{P}}(\mathfrak{g})$ be an orthonormal basis of $H_{\mathscr{P}}(\mathfrak{g})$. Since the expressions of interest are independent of the choice of orthonormal basis, we may assume with out loss of generality that $S_{\mathscr{P}}$ is a "good basis," i.e., $[h(s), h(\sigma)] = 0$ for all $s, \sigma \in [0, 1]$. (For example take the basis used in the proof of Lemma 5.9.) Then

$$\begin{aligned}
\mathrm{Ric}_{\mathscr{P}}\langle h, h\rangle &= \sum_{k \in S_{\mathscr{P}}} (R_{\mathscr{P}}\langle h, k\rangle\, k, h) \\
&= \sum_{k \in S_{\mathscr{P}}} (D^{\mathscr{P}}_h D^{\mathscr{P}}_k k - D^{\mathscr{P}}_k D^{\mathscr{P}}_h k - D^{\mathscr{P}}_{[h,k]_{\mathscr{P}}} k, h) \\
&= \sum_{k \in S_{\mathscr{P}}} (-D^{\mathscr{P}}_k D^{\mathscr{P}}_h k - D^{\mathscr{P}}_{[h,k]_{\mathscr{P}}} k, h) \\
&= \sum_{k \in S_{\mathscr{P}}} \{(D^{\mathscr{P}}_h k, D^{\mathscr{P}}_k h) - (D^{\mathscr{P}}_{[h,k]_{\mathscr{P}}} k, h)\} \\
&=: S - T.
\end{aligned}$$

Using Eq. (5.5), S may be written as

$$
\begin{aligned}
S &= \sum_{k\in S_{\mathscr{P}}}\sum_{i=0}^{n}\delta_i^{-1}\langle \delta_i D_h^{\mathscr{P}} k, \delta_i D_k^{\mathscr{P}} h\rangle \\
&= \sum_{k\in S_{\mathscr{P}}}\sum_{i=0}^{n}\delta_i^{-1}\left\langle [h_i^a,\delta_i k]-\delta_i\sum_{j=0}^{n}[h_j^a,\delta_j k],\right.\\
&\qquad\qquad \left.[k_i^a,\delta_i h]-\delta_i\sum_{j=0}^{n}[k_j^a,\delta_j h]\right\rangle \\
&= \sum_{k\in S_{\mathscr{P}}}\sum_{i=0}^{n}\left\{\delta_i^{-1}\langle [h_i^a,\delta_i k],[k_i^a,\delta_i h]\rangle-\sum_{j=0}^{n}\langle [h_i^a,\delta_i k],[k_j^a,\delta_j h]\rangle\right\} \\
&= \sum_{k\in S_{\mathscr{P}}}\sum_{i=0}^{n}\left\{\delta_i^{-1}\langle [h_i^a,\delta_i k],[k_i^a,\delta_i h]\rangle-\sum_{j=0}^{n}\langle [k_i^a,\delta_i h],[k_j^a,\delta_j h]\rangle\right\} \\
&= -\sum_{k\in S_{\mathscr{P}}}\sum_{i=0}^{n}\delta_i^{-1}\langle [k_i^a,[h_i^a,\delta_i k]],\delta_i h\rangle-\sum_{k\in S_{\mathscr{P}}}\left|\sum_{i=0}^{n}[k_i^a,\delta_i h]\right|^2 \\
&= -\sum_{k\in S_{\mathscr{P}}}\sum_{i=0}^{n}\delta_i^{-1}\langle ad_{k_i^a} ad_{h_i^a}\,\delta_i k,\delta_i h\rangle-\sum_{k\in S_{\mathscr{P}}}\left|\sum_{i=0}^{n}[k_i^a,\delta_i h]\right|^2,
\end{aligned}
$$

wherein the fourth equality we did a summation by parts. Namely we have used

$$\delta_i[h,k]=[h_i^a,\delta_i k]+[\delta_i h,k_i^a]=[h_i^a,\delta_i k]-[k_i^a,\delta_i h]$$

and the fact that $\sum_{i=0}^{n}\delta_i[h,k]=0$.

Similarly using Eq. (5.5) and $\sum_{i=0}^{n}\delta_i h=0$, T may be expressed as

$$
\begin{aligned}
T &= \sum_{k\in S_{\mathscr{P}}}\sum_{i=0}^{n}\delta_i^{-1}\left\{\left\langle [[h,k]_i^a,\delta_i k]-\delta_i\sum_{j=0}^{n}[[h,k]_j^a,\delta_j k],\delta_i h\right\rangle\right\} \\
&= \sum_{k\in S_{\mathscr{P}}}\sum_{i=0}^{n}\delta_i^{-1}\langle ad_{[h,k]_i^a}\,\delta_i k,\delta_i h\rangle.
\end{aligned}
$$

So combining the expressions for S and T shows

$$
\begin{aligned}
\mathrm{Ric}_{\mathscr{P}}\langle h,h\rangle &= S-T \\
&= -\sum_{k\in S_{\mathscr{P}}}\sum_{i=0}^{n}\delta_i^{-1}\langle ad_{k_i^a} ad_{h_i^a}\,\delta_i k+ad_{[h,k]_i^a}\,\delta_i k,\delta_i h\rangle \\
&\quad -\sum_{k\in S_{\mathscr{P}}}\left|\sum_{i=0}^{n}[k_i^a,\delta_i h]\right|^2.
\end{aligned}
$$

Now using the assumption that $S_{\mathscr{P}}$ is a good basis,

$$\begin{aligned} ad_{k_i^a} ad_{h_i^a} \delta_i k + ad_{[h,k]_i^a} \delta_i k &= [ad_{k_i^a}, ad_{h_i^a}] \delta_i k + ad_{[h,k]_i^a} \delta_i k \\ &= ad_{[k_i^a, h_i^a]} \delta_i k + ad_{[h,k]_i^a} \delta_i k \\ &= ad_{\{[h,k]_i^a - [h_i^a, k_i^a]\}} \delta_i k \\ &= \tfrac{1}{4} ad_{[\delta_i h, \delta_i k]} \delta_i k. \end{aligned}$$

Assembling the last two equations implies

$$\begin{aligned} \operatorname{Ric}_{\mathscr{P}}\langle h, h\rangle &= -\tfrac{1}{4} \sum_{k \in S_{\mathscr{P}}} \sum_{i=0}^{n} \delta_i^{-1} \langle ad_{[\delta_i h, \delta_i k]} \delta_i k, \delta_i h\rangle - \sum_{k \in S_{\mathscr{P}}} \left| \sum_{i=0}^{n} [k_i^a, \delta_i h] \right|^2 \\ &= \tfrac{1}{4} \sum_{k \in S_{\mathscr{P}}} \sum_{i=0}^{n} \delta_i^{-1} |[\delta_i h, \delta_i k]|^2 - \sum_{k \in S_{\mathscr{P}}} \left| \sum_{i=0}^{n} [k_i^a, \delta_i h] \right|^2. \end{aligned} \tag{5.7}$$

The above sums on $k \in S_{\mathscr{P}}$ may be computed using Lemma 5.9,

$$\begin{aligned} \tfrac{1}{4} \sum_{k \in S_{\mathscr{P}}} \sum_{i=0}^{n} \delta_i^{-1} |[\delta_i h, \delta_i k]|^2 &= \sum_{i=0}^{n} \delta_i^{-1} K\langle \delta_i h, \delta_i h\rangle \alpha_i \\ &= \int_0^1 \alpha_{\mathscr{P}}(\sigma) K\langle h'(\sigma), h'(\sigma)\rangle \, d\sigma, \end{aligned}$$

where

$$\alpha_i \equiv \{G_0(s_i, s_i) + G_0(s_{i+1}, s_{i+1}) - 2G_0(s_i, s_{i+1})\}/4$$

and

$$\alpha_{\mathscr{P}}(s) \equiv \sum_{i=0}^{n} \alpha_i 1_{(s_i, s_{i+1}]}(s). \tag{5.8}$$

Similarly,

$$\begin{aligned} \sum_{k \in S_{\mathscr{P}}} \left| \sum_{i=0}^{n} [k_i^a, \delta_i h] \right|^2 &= \sum_{i,j=0}^{n} K\langle \delta_i h, \delta_j h\rangle G_{ij} \\ &= \int_0^1 d\sigma \int_0^1 ds\, G_{\mathscr{P}}(\sigma, s) K\langle h'(s), h'(\sigma)\rangle, \end{aligned}$$

where $G_{\mathscr{P}}$ is defined in Eq. (5.6). Hence

$$\begin{aligned} \operatorname{Ric}_{\mathscr{P}}\langle h, h\rangle &= \int_0^1 \alpha_{\mathscr{P}}(\sigma) K\langle h'(\sigma), h'(\sigma)\rangle \, d\sigma \\ &\quad - \int_0^1 d\sigma \int_0^1 ds\, G_{\mathscr{P}}(\sigma, s) K\langle h'(s), h'(\sigma)\rangle. \end{aligned} \tag{5.9}$$

The polarization of Eq. (5.9) and Eq. (2.2) shows

$$\begin{aligned}\operatorname{Ric}_{\mathscr{P}}\langle h, k\rangle - \operatorname{Ric}\langle h, k\rangle &= \int_0^1 \alpha_{\mathscr{P}}(\sigma)\, K\langle h'(\sigma), k'(\sigma)\rangle\, d\sigma \\ &\quad - \int_0^1 d\sigma \int_0^1 ds \{G_{\mathscr{P}}(\sigma, s) - G_0(\sigma, s)\}\, K\langle h'(s), k'(\sigma)\rangle\end{aligned}$$

for all $h, k \in H_{\mathscr{P}}(\mathfrak{g})$.

Let $\|\cdot\|_u$ denote the supremum norm on functions, then it easily follows from the last equation, for all $h, k \in H_{\mathscr{P}}(\mathfrak{g})$, that

$$\begin{aligned}&|((\operatorname{Ric} - \operatorname{Ric}_{\mathscr{P}})\, h, k)| \\ &\quad\leqslant C\, \|\alpha_{\mathscr{P}}\|_u\, \|h\|\, \|k\| + \int_0^1 d\sigma \int_0^1 ds\, |G_0(\sigma, s) - G_{\mathscr{P}}(\sigma, s)|\, |K\langle h'(s), k'(\sigma)\rangle| \\ &\quad\leqslant C(\|\alpha_{\mathscr{P}}\|_u + \|G_0 - G_{\mathscr{P}}\|_u)\, \|h\|\, \|k\|.\end{aligned}$$

Hence

$$\|P_{\mathscr{P}}(\operatorname{Ric} - \operatorname{Ric}_{\mathscr{P}})\, P_{\mathscr{P}}\|_{op} \leqslant C(\|\alpha_{\mathscr{P}}\|_u + \|G_0 - G_{\mathscr{P}}\|_u). \tag{5.10}$$

Looking at the definitions of $\alpha_{\mathscr{P}}$ and $G_{\mathscr{P}}$ and using uniform continuity of G_0, it is easily seen that $\|\alpha_{\mathscr{P}}\|_u + \|G_0 - G_{\mathscr{P}}\|_u \to 0$ as $|\mathscr{P}| \to 0$. This observation and Eq. (5.10) finishes the proof. Q.E.D.

6. INTEGRATION BY PARTS ON THE LOOP GROUP

In this section we will prove Theorem 1.5 by passing to the limit in Theorem 5.3. Before doing this it is first convenient to rewrite Eq. (5.1) as

$$\begin{aligned}&E[(\tilde{h}f)(\Sigma(T))] \\ &\quad= E\left[f(\Sigma(T)) \int_0^T (U_{\mathscr{P}}(t)\, U_{\mathscr{P}}(T)^{-1}\, h_{\mathscr{P}}, (\dot{l}(t) - \tfrac{1}{2}\, l(t)\, \operatorname{Ric}_{\mathscr{P}})\, \overleftarrow{d\beta}_{\mathscr{P}}(t)) \right].\end{aligned}$$

Setting $U_{\mathscr{P}}(t, T) \equiv U_{\mathscr{P}}(t)\, U_{\mathscr{P}}(T)^{-1}$, $H_{\mathscr{P}}(t) \equiv U_{\mathscr{P}}(t, T)\, h_{\mathscr{P}} = U_{\mathscr{P}}(t)\, U_{\mathscr{P}}(T)^{-1}\, h_{\mathscr{P}}$, we see that this last equation may be written as

$$E[(\tilde{h}f)(\Sigma(T))] = E\left[f(\Sigma(T)) \int_0^T (H_{\mathscr{P}}(t), (\dot{l}(t) - \tfrac{1}{2}\, l(t)\, \mathrm{Ric}_{\mathscr{P}})\, \overleftarrow{d\beta}_{\mathscr{P}}(t)) \right], \tag{6.1}$$

where $H_{\mathscr{P}}(t)$ is the solution to the Stratonovich stochastic integral equation

$$h - H_{\mathscr{P}}(t) + \int_t^T D^{\mathscr{P}}_{\delta\beta_{\mathscr{P}}(\tau)} H_{\mathscr{P}}(\tau) = 0. \tag{6.2}$$

The reader should notice that the process $H_{\mathscr{P}}$ is not adapted to the filtration $\{\mathscr{F}_t\}_{t \geqslant 0}$. Nevertheless the integral in (6.2) may be defined as the usual L^2 limit of Riemann sum approximations of the form

$$\int_t^T D^{\mathscr{P}}_{\delta\beta_{\mathscr{P}}(\tau)} H_{\mathscr{P}} \equiv \lim_{|\pi| \to 0} \tfrac{1}{2} \sum_{\tau \in \pi} D^{\mathscr{P}}_{(\beta(\tau+) - \beta(\tau))} (H_{\mathscr{P}}(\tau) + H_{\mathscr{P}}(\tau+)), \tag{6.3}$$

where π denotes a partition of $[t, T]$, for $\tau \in \pi$, $\tau+$ denotes the successor to τ in π, and $|\pi|$ is the mesh size of the partition. To show the convergence, notice from the usual adapted theory that

$$\int_t^T D^{\mathscr{P}}_{\delta\beta_{\mathscr{P}}(\tau)} U_{\mathscr{P}}(\tau) \equiv \lim_{|\pi| \to 0} \tfrac{1}{2} \sum_{\tau \in \pi} D^{\mathscr{P}}_{(\beta(\tau+) - \beta(\tau))} (U_{\mathscr{P}}(\tau) + U_{\mathscr{P}}(\tau+)), \tag{6.4}$$

where the sum exists in L^2. Since Eq. (6.3) is obtained from Eq. (6.4) by multiplication on the right by $U_{\mathscr{P}}^{-1}(T)\, h_{\mathscr{P}}$, Eq. (6.3) has the same convergence properties as Eq. (6.4). Moreover, by the discussion in Section 4.1 in [5], the integrals in Eq. (6.3) and (6.4) may be expressed in terms of Backwards Itô integrals,

$$\begin{aligned}
\int_t^T D^{\mathscr{P}}_{\delta\beta_{\mathscr{P}}(\tau)} H_{\mathscr{P}}(\tau) &= \int_t^T D^{\mathscr{P}}_{\overleftarrow{d\beta}_{\mathscr{P}}(\tau)} H_{\mathscr{P}}(\tau) - \tfrac{1}{2} \int_t^T D^{\mathscr{P}}_{d\beta_{\mathscr{P}}(\tau)}\, dH_{\mathscr{P}}(\tau) \\
&= \int_t^T D^{\mathscr{P}}_{\overleftarrow{d\beta}_{\mathscr{P}}(\tau)} H_{\mathscr{P}}(\tau) + \tfrac{1}{2} \sum_{h \in S_{\mathscr{P}}} \int_t^T D^{\mathscr{P}}_h D^{\mathscr{P}}_h H_{\mathscr{P}}(\tau)\, d\tau \\
&= \int_t^T D^{\mathscr{P}}_{\overleftarrow{d\beta}_{\mathscr{P}}(\tau)} H_{\mathscr{P}}(\tau) + \tfrac{1}{2} \int_t^T \Delta^{(1)}_{\mathscr{P}} H_{\mathscr{P}}(\tau)\, d\tau,
\end{aligned}$$

where

$$\int_t^T D^{\mathscr{P}}_{\overleftarrow{d\beta}_{\mathscr{P}}(\tau)} H_{\mathscr{P}}(\tau) \equiv \lim_{|\pi| \to 0} \sum_{\tau \in \pi} D^{\mathscr{P}}_{(\beta(\tau+) - \beta(\tau))} H_{\mathscr{P}}(\tau+),$$

Therefore (6.2) is equivalent to the backwards stochastic integral equation,

$$h - H_{\mathscr{P}}(t) + \int_t^T D^{\mathscr{P}}_{\overleftarrow{d\beta_{\mathscr{P}}(\tau)}} H_{\mathscr{P}}(\tau) + \tfrac{1}{2} \int_t^T \Delta^{(1)}_{\mathscr{P}} H_{\mathscr{P}}(\tau)\, d\tau = 0. \tag{6.5}$$

Following the notation and discussion in Section 8.3 of the Appendix below, let $\beta^T(t) \equiv \beta(T-t) - \beta(T)$ and $H^T_{\mathscr{P}}(t) \equiv H_{\mathscr{P}}(T-t)$ for $t \in [0, T]$ and $\{\mathscr{F}^T_t\}_{t \in [0, T]}$ denote the filtration generated by $\{\beta^T(t)\}_{t \in [0, T]}$ appropriately completed. Then $\{\beta^T(t)\}_{t \in [0, T]}$ is again a standard $\{\mathscr{F}^T_t\}_{t \in [0, T]}$-Brownian motion and Eq. (6.5) may be expressed as

$$h - H_{\mathscr{P}}(t) - \int_0^{T-t} D^{\mathscr{P}}_{d\beta^T_{\mathscr{P}}(\tau)} H^T_{\mathscr{P}}(\tau) + \tfrac{1}{2} \int_0^{T-t} \Delta^{(1)}_{\mathscr{P}} H^T_{\mathscr{P}}(\tau)\, d\tau = 0.$$

Replacing t by $T-t$ in this last equation shows that Eq. (6.5) (for $H_{\mathscr{P}}$) is equivalent to the following standard forward stochastic differential equation (for $H^T_{\mathscr{P}}$):

$$H^T_{\mathscr{P}}(t) = h - \int_0^t D^{\mathscr{P}}_{d\beta^T_{\mathscr{P}}(\tau)} H^T_{\mathscr{P}}(\tau) + \tfrac{1}{2} \int_0^t \Delta^{(1)}_{\mathscr{P}} H^T_{\mathscr{P}}(\tau)\, d\tau. \tag{6.6}$$

This last equation may be written in differential form as

$$dH^T_{\mathscr{P}}(t) + D^{\mathscr{P}}_{d\beta^T_{\mathscr{P}}(t)} H^T_{\mathscr{P}}(t) - \tfrac{1}{2} \Delta^{(1)}_{\mathscr{P}} H^T_{\mathscr{P}}(\tau) = 0 \qquad \text{with} \quad H^T_{\mathscr{P}}(0) = h. \tag{6.7}$$

In analogy to Eq. (6.5), for $h \in H_0(\mathfrak{g})$, we let $H(t)$ denote the solution to the backwards stochastic differential equation

$$h - H(t) + \int_t^T D_{\overleftarrow{d\beta(\tau)}} H(\tau) + \tfrac{1}{2} \int_t^T \Delta^{(1)} H(\tau)\, d\tau = 0. \tag{6.8}$$

THEOREM 6.1 (Backwards Parallel Translation). *Given $T > 0$ and $h \in H_0(\mathfrak{g})$, there exists a unique $H_0(\mathfrak{g})$-valued continuous backwards semi-martingale $H(t)$, relative to the filtration $\{\mathscr{F}^T_t\}_{t \in [0, T]}$, solving Eq. (6.8). Moreover there exists a process $t \in [0, T] \to U(t, T) \in O(H_0(\mathfrak{g}))$ such that for all $h \in H_0(\mathfrak{g})$, $H(t) \equiv U(t, T)\, h$ is the unique solution to Eq. (6.8).*

Proof. Using Definition 8.6 of the backwards stochastic integral in Section 8.3 of the Appendix and the same argument used above in passing from Eq. (6.5) to Eq. (6.7), we find that Eq. (6.8) is equivalent to

$$H^T(t) = h - \int_0^t D_{d\beta^T(\tau)} H^T(\tau) + \tfrac{1}{2} \int_0^t \Delta^{(1)} H^T(\tau)\, d\tau, \tag{6.9}$$

where $H^T \equiv H(T-t)$. With this observation, the theorem follows from Theorem 4.1 and Lemma 4.3 above. Q.E.D.

THEOREM 6.2 (Integration by Parts). *Let* $T>0$, $l \in H(\mathbb{R})$ *such that* $l(T)=1$, $h \in H_0(\mathfrak{g})$, *and* $H(t)$ (*for* $t \in [0, T]$) *be the unique solution to Eq.* (6.8). *Then for all* $f \in \mathscr{F}C^\infty(\mathscr{L})$,

$$E[(\tilde{h}f)(\Sigma_T)] = E\left[f(\Sigma_T) \int_0^T (\{\dot{l}(t) - \tfrac{1}{2} l(t) \operatorname{Ric}\} H(t), \overleftarrow{d\beta}(t))\right], \quad (6.10)$$

where $\overleftarrow{d\beta}$ *denotes the backwards stochastic differential. In particular, Eq.* (1.14) *of Theorem* 1.5 *follows from Eq.* (6.10) *by choosing* $l(t) \equiv t/T$.

Remark 6.3. The backwards stochastic integral appearing in Eq. (6.10) is well defined and we have the estimate

$$Ez^2 := E \int_0^T \|\{\dot{l}(t) - \tfrac{1}{2} l(t) \operatorname{Ric}\} H(t)\|^2 \, dt$$

$$\leqslant \|h\|^2 \cdot \int_0^T [\,|\dot{l}(t)| + \tfrac{1}{2} |l(t)| \, \|\operatorname{Ric}\|_{op}]^2 \, dt < \infty.$$

More generally, using Burkholder's inequality, for all $p \in [2, \infty)$ there are constants $C_p < \infty$ such that

$$\|z\|_{L^p(P)} \leqslant C_p \|h\| \left(\int_0^T [\,|\dot{l}(t)| + \tfrac{1}{2} |l(t)| \, \|\operatorname{Ric}\|_{op}]^2 \, dt \right)^{1/2} < \infty.$$

6.1. *Passing to the Limit*

The rest of this section will now be devoted to the proof of Theorem 6.2 which will be carried out by letting $|\mathscr{P}| \to 0$ in Eq. (6.1). The following theorem is the key result needed to take this limit. Recall the notation used above Eq. (6.1), namely $U_{\mathscr{P}}(t, T) \equiv U_{\mathscr{P}}(t)\, U_{\mathscr{P}}(T)^{-1}$ and $H_{\mathscr{P}}(t) \equiv U_{\mathscr{P}}(t, T)\, h_{\mathscr{P}} = U_{\mathscr{P}}(t)\, U_{\mathscr{P}}(T)^{-1}\, h_{\mathscr{P}}$.

THEOREM 6.4. *Let* $T \in (0, \infty)$, $h \in H_{\mathscr{P}_0}(\mathfrak{g})$, *and* $\mathscr{P}_n$ *be a sequence of partitions of* $[0, 1]$ *such that* $\mathscr{P}_{n+1} \supset \mathscr{P}_n$ *for all n and* $\lim_{n\to\infty} |\mathscr{P}_n| = 0$. *Set*

$$z \equiv \int_0^T (\{\dot{l}(t) - \tfrac{1}{2} l(t) \operatorname{Ric}\}\, U(t, T)\, h, \overleftarrow{d\beta}(t)), \quad (6.11)$$

and

$$z_n \equiv \int_0^T (\{\dot{l}(t) - \tfrac{1}{2}\, l(t)\, \mathrm{Ric}_{\mathscr{P}_n}(t, T)\, h, \overleftarrow{d\beta}_{\mathscr{P}_n}(t)). \tag{6.12}$$

Then z_n *converges to* z *in* L^2.

Proof. To simplify notation let $P_n \equiv P_{\mathscr{P}_n}\colon H_0(\mathfrak{g}) \to H_0(\mathfrak{g})$ denote orthogonal projection onto $H_{\mathscr{P}_n}(\mathfrak{g})$, $\mathrm{Ric}_n \equiv \mathrm{Ric}_{\mathscr{P}_n}$, and $U_n \equiv U_{\mathscr{P}_n}$. We will first show that $P_n \to I$ strongly as $n \to \infty$. To prove this it suffices to show, since $\mathrm{ran}(P_{n+1}) \supset \mathrm{ran}(P_n)$ for all n, that $D \equiv \bigcup_n \mathrm{ran}(P_n)$ is dense in $H_0(\mathfrak{g})$. To see that D is dense, first notice that $G_0(s, \cdot)\, A \in D$ for all $s \in Q \equiv \bigcup_n \mathscr{P}_n$ and $A \in \mathfrak{g}$. Hence if $h \perp D$, then $\langle h(s), A \rangle = (h, G_0(s, \cdot)\, A) = 0$ for all $s \in Q$ and $A \in \mathfrak{g}$. Since h is continuous and $Q \subset [0, 1]$ is dense, it follows that $h \equiv 0$. Therefore D is dense in $H_0(\mathfrak{g})$ and hence $P_n \to I$ strongly.

In the remainder of the proof, ε_n will be used to denote any generic sequence of non-negative real numbers such that $\lim_{n \to \infty} \varepsilon_n = 0$. (The value of ε_n may vary from line to line in the following proof, but in all cases $\lim_{n \to \infty} \varepsilon_n = 0$.)

Using the isometry property of the Itô Integral,

$$\begin{aligned}
E\,|z - z_n|^2 &= E\,\bigg| \int_0^T (\{\dot{l}(t) - \tfrac{1}{2}\, l(t)\, \mathrm{Ric}\}\; U(t, T)\, h \\
&\qquad - \{\dot{l}(t) - \tfrac{1}{2}\, l(t)\, \mathrm{Ric}_n\}\; U_n(t, T)\, h, \overleftarrow{d\beta}(t))\bigg|^2 \\
&= E \int_0^T \|\{\dot{l}(t) - \tfrac{1}{2}\, l(t)\, \mathrm{Ric}\}\; U(t, T)\, h \\
&\qquad - \{\dot{l}(t) - \tfrac{1}{2}\, l(t)\, \mathrm{Ric}_n\}\; U_n(t, T)\, h\|^2\, dt \\
&\leqslant 2 \int_0^T E\, \|P_n\{\dot{l}(t) - \tfrac{1}{2}\, l(t)\, \mathrm{Ric}\}\; U(t, T)\, h \\
&\qquad - \{\dot{l}(t) - \tfrac{1}{2}\, l(t)\, \mathrm{Ric}_n\}\; U_n(t, T)\, h\|^2\, dt \\
&\qquad + 2 \int_0^T E\, \|(I - P_n)\{\dot{l}(t) - \tfrac{1}{2}\, l(t)\, \mathrm{Ric}\}\; U(t, T)\, h\|^2\, dt.
\end{aligned}$$

Because $P_n \to I_{H_0(\mathfrak{g})}$ strongly as $n \to \infty$, $U(t, T)$ is unitary and Ric is a bounded operator, we may apply the dominated convergence theorem to find

$$\lim_{n\to\infty} \int_0^T E\,\|(I-P_n)\{\dot{l}(t) - \tfrac{1}{2}\, l(t)\,\mathrm{Ric}\}\, U(t,T)\, h\|^2\, dt = 0. \tag{6.13}$$

The last two displayed equations imply that

$$\begin{aligned} E\,|z-z_n|^2 \leqslant 4 \int_0^T \dot{l}^2(t)\, E\,\|\{P_n U(t,T) - U_n(t,T)\}\, h\|^2\, dt \\ + \int_0^T l^2(t)\, E\,\|\{P_n\,\mathrm{Ric}\, U(t,T) - \mathrm{Ric}_n\, U_n(t,T)\}\, h\|^2\, dt + \varepsilon_n. \end{aligned} \tag{6.14}$$

As in the proof of Eq. (6.13),

$$\begin{aligned} \limsup_{n\to\infty} \int_0^T l^2(t)\, E\,\|P_n\,\mathrm{Ric}(I-P_n)\, U(t,T)\, h\|^2\, dt \\ \leqslant \limsup_{n\to\infty} \int_0^T l^2(t)\, \|\mathrm{Ric}\|^2\, E\,\|(I-P_n)\, U(t,T)\, h\|^2\, dt = 0, \end{aligned}$$

which along with Eq. (6.14) implies that

$$\begin{aligned} E\,|z-z_n|^2 \leqslant 4 \int_0^T \dot{l}^2(t)\, E\,\|\{P_n U(t,T) - U_n(t,T)\}\, h\|^2\, dt \\ + 2 \int_0^T l^2(t)\, E\,\|\{P_n\,\mathrm{Ric}\, P_n U(t,T) - \mathrm{Ric}_n U_n(t,T)\}\, h\|^2\, dt + \varepsilon_n. \end{aligned} \tag{6.15}$$

Recall that Theorem 5.10 asserts that

$$\lim_{n\to\infty} \|P_n\,\mathrm{Ric}\, P_n - \mathrm{Ric}_n\, P_n\|_{op}^2 = 0$$

and in particular this implies that $C \equiv \sup_n \|\mathrm{Ric}_n P_n\|_{op}^2 < \infty$. Therefore

$$\begin{aligned}
&\int_0^T l^2(t)\, E\, \|\{P_n \,\mathrm{Ric}\, P_n U(t,T) - \mathrm{Ric}_n U_n(t,T)\}\, h\|^2\, dt \\
&\quad \leqslant 2 \int_0^T l^2(t)\, \|P_n \,\mathrm{Ric}\, P_n - \mathrm{Ric}_n P_n\|_{op}^2\, E\, \|U(t,T)\, h\|^2\, dt \\
&\qquad + 2 \int_0^T l^2(t)\, E\, \|\mathrm{Ric}_n\{P_n U(t,T) - U_n(t,T)\}\, h\|^2\, dt \\
&\quad \leqslant 2\, \|P_n \,\mathrm{Ric}\, P_n - \mathrm{Ric}_n P_n\|_{op}^2\, \|h\|^2 \cdot \int_0^T l^2(t)\, dt \\
&\qquad + 2 \int_0^T l^2(t)\, \|\mathrm{Ric}_n P_n\|_{op}^2\, E\, \|\{P_n U(t,T) - U_n(t,T)\}\, h\|^2\, dt \\
&\quad \leqslant 2C \int_0^T l^2(t)\, E\, \|\{P_n U(t,T) - U_n(t,T)\}\, h\|^2\, dt + \varepsilon_n.
\end{aligned}$$

Using this estimate in Eq. (6.15) gives

$$\begin{aligned}
E\,|z - z_n|^2 \leqslant 4 \int_0^T (\dot{l}^2(t) + C l^2(t)) \\
\times E\, \|\{P_n U(t,T) - U_n(t,T)\}\, h\|^2\, dt + \varepsilon_n. \qquad (6.16)
\end{aligned}$$

Since $\|\{P_n U(t,T) - U_n(t,T)\}\, h\| \leqslant 2\, \|h\|$, the theorem follows from Eq. (6.16) and the dominated convergence theorem provided that

$$\lim_{n \to \infty} E \|\{P_n U(t,T) - U_n(t,T)\}\, h\|^2 = 0.$$

This is the content of the next lemma. Q.E.D.

LEMMA 6.5. *Keeping the notation of the previous theorem,*

$$\lim_{n \to \infty} \sup_{t \in [0,T]} E\, \|\{P_n U(t,T) - U_n(t,T)\}\, h\|^2 = 0.$$

Proof. Recall that β is the $\mathscr{L}(\mathfrak{g})$-valued Brownian motion described in Section 3.1. Let $\beta_n \equiv P_n \beta = \beta_{\mathscr{P}_n}$, $D^n \equiv D^{\mathscr{P}_n}$, $\Delta_n^{(1)} \equiv \Delta_{\mathscr{P}_n}^{(1)}$, $Q_n \equiv I - P_n$, $H(t) \equiv U(t,T)\, h$, $H_n(t) \equiv U_n(t,T)\, h$, and $\alpha_n(t) \equiv H_n(t) - P_n H(t)$. By Eq. (6.8),

$$dH = -D_{\overleftarrow{d\beta}} H - \tfrac{1}{2} \Delta^{(1)} H\, dt = 0 \qquad \text{with} \quad H(T) = h$$

and Eq. (6.5)

$$dH_n = -D_{\overleftarrow{d\beta}_n} H_n - \tfrac{1}{2}\Delta_n^{(1)} H_n\, dt = 0 \qquad \text{with} \quad H_n(T) = h.$$

Therefore,

$$\begin{aligned}
-d\alpha_n &= -dH_n + P_n\, dH \\
&= D^n_{\overleftarrow{d\beta}_n} H_n - P_n D_{\overleftarrow{d\beta}} H + \tfrac{1}{2}(\Delta_n^{(1)} H_n - P_n \Delta^{(1)} H)\, dt \\
&= P_n D_{\overleftarrow{d\beta}_n}(H_n - H) - P_n D_{Q_n \overleftarrow{d\beta}} H \\
&\quad + \tfrac{1}{2}[\Delta_n^{(1)}(\alpha_n + P_n H) - P_n \Delta^{(1)}(P_n H + Q_n H)]\, dt \\
&= P_n D_{\overleftarrow{d\beta}_n}\alpha_n - P_n D_{\overleftarrow{d\beta}_n} Q_n H - P_n D_{Q_n \overleftarrow{d\beta}} H \\
&\quad + \tfrac{1}{2}\Delta_n^{(1)}\alpha_n\, dt + \tfrac{1}{2}(\Delta_n^{(1)} - P_n \Delta^{(1)} P_n)\, P_n H\, dt - \tfrac{1}{2} P_n \Delta^{(1)} Q_n H\, dt,
\end{aligned}$$

with $\alpha_n(T) = 0$. More precisely we have

$$\alpha_n(t) = A_n(t) - B_n(t) - C_n(t) + \tfrac{1}{2}(D_n(t) + E_n(t) - F_n(t)),$$

where

$$A_n(t) \equiv \int_t^T P_n D_{\overleftarrow{d\beta}_n}\alpha_n, \qquad B_n(t) \equiv \int_t^T P_n D_{\overleftarrow{d\beta}_n} Q_n H,$$

$$C_n(t) \equiv \int_t^T P_n D_{Q_n \overleftarrow{d\beta}} H, \qquad D_n(t) \equiv \int_t^T \Delta_n^{(1)}\alpha_n(\tau)\, d\tau,$$

$$E_n(t) \equiv \int_t^T (\Delta_n^{(1)} - P_n \Delta^{(1)} P_n)\, P_n H(\tau)\, d\tau,$$

and

$$F_n(t) \equiv \int_t^T P_n \Delta^{(1)} Q_n H(\tau)\, d\tau.$$

Let us now estimate the L^2-norms of the four terms not containing α_n.

B_n-term,

$$\begin{aligned}
E\,\|B_n(t)\|^2 &= E\int_t^T \sum_{l \in S_0} \|P_n D_{P_n l} Q_n H(u)\|^2\, du \\
&\leqslant E\int_0^T \sum_{l \in S_0} \|D_l Q_n H(u)\|^2\, du \\
&\leqslant E\int_0^T \|D\|_{op}^2\, \|Q_n H(u)\|^2\, du,
\end{aligned}$$

which tends to zero by the dominated convergence theorem and the fact that Q_n converges strongly to zero.

C_n-term,

$$E\,\|C_n(t)\|^2 = E\int_t^T \sum_{l\in S_0} \|P_n\, D_{Q_n l} H(u)\|^2\,du$$
$$\leqslant E\int_0^T \sum_{l\in S_0} \|D_{Q_n l} H(u)\|^2\,du$$
$$= E\int_0^T \|(Q_n^{tr}\otimes I)\,DH(u)\|^2\,du, \tag{6.17}$$

where $Q_n^{tr}\colon H_0(\mathfrak{g})^* \to H_0(\mathfrak{g})^*$ is the transpose of Q_n. Since Q_n is an orthogonal projection operator, it easily follows that Q_n is unitarily equivalent to Q_n^{tr} under the natural unitary isomorphism between $H_0(\mathfrak{g})$ and its dual $H_0(\mathfrak{g})^*$. In particular $Q_n^{tr} \to 0$ strongly as $n\to\infty$ and hence $(Q_n^{tr}\otimes I)\to 0$ strongly as $n\to\infty$, see the proposition on p. 299 of Reed and Simon [23]. So again by the dominated convergence theorem, it follows from Eq. (6.17) that $\lim_{n\to\infty} E\,\|C_n(t)\|^2 = 0$.

E_n-term,

$$E\,\|E_n(t)\|^2 \leqslant (T-t)\,\|\Delta_n^{(1)} - P_n\,\Delta^{(1)} P_n\|_{op}^2\, E\int_t^T \|H(u)\|^2\,du$$
$$= T\,\|\Delta_n^{(1)} - P_n\,\Delta^{(1)} P_n\|_{op}^2 \int_t^T \|h\|^2\,du$$
$$\leqslant T^2\,\|\Delta_n^{(1)} - P_n\,\Delta^{(1)} P_n\|_{op}^2\,\|h\|^2,$$

which tends to zero as $n\to\infty$ by Theorem 5.8.

F_n-term,

$$E\,\|F_n(t)\|^2 \leqslant (T-t)\,\|P_n\,\Delta^{(1)}\|_{op}^2\, E\int_t^T \|Q_n H(u)\|^2\,du$$
$$\leqslant T\,\|\Delta^{(1)}\|_{op}^2\, E\int_0^T \|Q_n H(u)\|^2\,du,$$

which again tends to zero as $n\to\infty$ because of the dominated convergence theorem and the fact that Q_n is strongly convergent to the zero.

Combining the above four estimates with the expression for α yields

$$E\,\|\alpha_n(t)\|^2 \leqslant 2E\,\|A_n(t)+D_n(t)\|^2+\varepsilon_n \leqslant 4E\,\|A_n(t)\|^2+4E\,\|D_n(t)\|^2+\varepsilon_n$$

$$\leqslant 4E\left\|\int_t^T P_n\,D_{\overleftarrow{d\beta}_n}\alpha_n\right\|^2+E\left\|\int_t^T \varDelta_n^{(1)}\alpha_n(u)\,du\right\|^2+\varepsilon_n$$

$$=4\sum_{l\in S_0}\int_t^T E\,\|P_n\,D_{P_n l}\alpha_n(u)\|^2\,du+E\left\|\int_t^T \varDelta_n^{(1)}\alpha_n(u)\,du\right\|^2+\varepsilon_n$$

$$\leqslant 4\,\|D\|_{op}^2\int_t^T E\,\|\alpha_n(u)\|^2\,du+(T-t)\,E\int_t^T \|\varDelta_n^{(1)}\alpha_n(u)\|^2\,du+\varepsilon_n$$

$$\leqslant (4\,\|D\|_{op}^2+T\sup_n \|\varDelta_n^{(1)}\|_{op}^2)\,E\int_t^T \|\alpha_n(u)\|^2\,du+\varepsilon_n,$$

where ε_n denotes a generic sequence of positive numbers with $\lim_{n\to\infty}\varepsilon_n=0$. By Theorem 5.8, $\sup_n \|\varDelta_n^{(1)}\|_{op}^2<\infty$ and hence the proof of the Lemma may be concluded with an application of Gronwall's inequality. Q.E.D.

Proof of Theorem 6.2. Let $t>0$ and $f\in\mathscr{F}C^\infty(\mathscr{L})$. Choose a partition $\mathscr{P}$ of $[0, 1]$ so that $f=F\circ\pi_{\mathscr{P}}$ for some C^1-function F on $G^{\mathscr{P}}$. Let $\mathscr{P}_0$ be a partition which refines $\mathscr{P}$ (i.e., $\mathscr{P}\subset\mathscr{P}_0$) and for the moment assume that $h\in H_{\mathscr{P}_0}(\mathfrak{g})$.

Let $\mathscr{P}_n$ be a sequence of partitions such that $\mathscr{P}_0\subset\mathscr{P}_n\subset\mathscr{P}_{n+1}$ for all $n=1, 2, ...$ Let z and z_n be the random variables as in Theorem 6.4, see Eqs. (6.11) and (6.12). By Eq. (6.1), with $\mathscr{P}$ replaced by $\mathscr{P}_n$,

$$E[(\tilde{h}f)(\Sigma_T)]=E[f(\Sigma_T)\,z_n] \tag{6.18}$$

holds for all n. By Theorem 6.4, we may let $n\to\infty$ in (6.18) to conclude that

$$E[(\tilde{h}f)(\Sigma_T)]=E[f(\Sigma_T)\,z]$$
$$=E\left[f(\Sigma_T)\int_0^T (\{\dot{l}(t)-\tfrac{1}{2}\,l(t)\,\mathrm{Ric}\}\;U(t, T)\,h,\,\overleftarrow{d\beta}(t))\right]. \tag{6.19}$$

By Remark 6.3 and the fact that f is bounded, the right hand side of (6.19) is continuous in $h\in H_0(\mathfrak{g})$. Similarly, since the $H_0(\mathfrak{g})$ norm is stronger than the supremum norm and df is bounded, it follows that the left-hand-side of Eq. (6.18) is also continuous in $h\in H_0(\mathfrak{g})$. The continuity of both sides of Eq. (6.19), coupled with the fact that the span of the union of $H_{\mathscr{P}_0}(\mathfrak{g})$ over all finite partitions $\mathscr{P}_0$ of $[0, 1]$ which refine $\mathscr{P}$ is dense in $H_0(\mathfrak{g})$, implies that (6.19) is valid for all $h\in H_0(\mathfrak{g})$. Q.E.D.

COROLLARY 6.6. *For each* $h \in H_0(\mathfrak{g})$, *the differential operator* $\tilde{h}$ *with domain* $\mathscr{F}C^\infty(\mathscr{L}(G))$ *is a densely defined closable operator on* $L^2(\mathscr{L}(G), \nu_T)$. *Moreover the* L^2-*adjoint of* $\tilde{h}^*$ *of* $\tilde{h}$ *satisfies*

$$\tilde{h}^* |_{\mathscr{F}C^\infty(\mathscr{L}(G))} = -\tilde{h} + \alpha_h, \tag{6.20}$$

where $\alpha_h\colon \mathscr{L}(G) \to \mathbb{R}$ *is a Borel measurable function such that*

$$\alpha_h(\Sigma_T) \equiv \frac{1}{T} E\left[\int_0^T \left(\left\{ I - \frac{1}{2}\tau \operatorname{Ric} \right\} H(\tau), \overleftarrow{d\beta}(\tau) \right) \middle| \sigma(\Sigma_T) \right] \qquad a.s. \tag{6.21}$$

Proof. Let $u, v \in \mathscr{F}C^\infty(\mathscr{L})$ and $l(t) = t/T$. Then apply Theorem 6.2 with $f \in \mathscr{F}C^\infty(\mathscr{L})$ replaced by $uv \in \mathscr{F}C^\infty(\mathscr{L})$ to find

$$E[(\tilde{h}u)(\Sigma_T)\, v(\Sigma_T) + u(\Sigma_T)(\tilde{h}v)(\Sigma_T)] = E[u(\Sigma_T)\, \alpha_h(\Sigma_T)]$$

or equivalently

$$\int_{\mathscr{L}(G)} (\tilde{h}u)(g)\, v(g)\, d\nu_T(g) = \int_{\mathscr{L}(G)} u(g)[-(\tilde{h}v)(g) + \alpha_h(g)\, v(g)]\, d\nu_T(g).$$

This proves Eq. (6.20) and the fact that $\tilde{h}$ is closable, since the properties of being closable and having a densely defined adjoint are equivalent.

Q.E.D.

7. QUASI-INVARIANCE OF THE HEAT KERNEL MEASURE

In this section we will show that the measure $\nu_T \equiv \operatorname{Law}(\Sigma_T)$ is quasi-invariant under both right and left translations by finite energy paths in $\mathscr{L}(G)$, see Corollary 7.7 and 7.10 below. Our method will be modeled on a technique in Cruzeiro [3] (see also Dennis Bell [1] and Gunnar Peters [20, 21]) for proving quasi-invariance of flows of certain vector fields on an abstract Wiener space. In order to carry out the proof it is necessary to recall a few results from the finite dimensional case.

7.1. *Finite Dimensional Preliminaries*

Let M be a finite dimensional manifold and X be a smooth complete vector field on M. We will denote the flow of X by $\{e^{tX}\}_{t \in \mathbb{R}}$ so that $e^{tX}\colon M \to M$ is a diffeomorphism for all $t \in \mathbb{R}$ such that $e^{0X} = id_M$ and $de^{tX}/dt = X \circ e^{tX}$. Suppose that σ is Borel measure on M such that, in every coordinate chart, σ has a smooth positive density relative to Lebesgue measure. Then the standard change of variable theorem guarantees that

$e_*^{tX}\sigma \equiv \sigma \circ e^{-tX}$ is absolutely continuous relative to σ and that the Radon-Nikodym derivative $Z_t(m) \equiv (de_*^{tX}\sigma/d\sigma)(m)$ may be chosen to be a smooth positive function of $(t, m) \in \mathbb{R} \times M$.

DEFINITION 7.1 (Divergence). *The divergence* $\operatorname{div}_\sigma(X)$ *of* X *relative to* σ *may be defined as*

$$\operatorname{div}_\sigma(X) = -\frac{d}{dt}\Big|_0 Z_t. \tag{7.1}$$

(*The reason for the minus sign is to adhere to the standard sign conventions for the divergence defined by other means.*)

The following proposition summarizes some well known properties of Z_t and $\operatorname{div}_\sigma(X)$.

PROPOSITION 7.2. *Let* X, Z_t, *and* $\operatorname{div}_\sigma(X)$ *be as above.*

1. *Suppose that* $B \subset M$ *is a Borel subset of* M *such that* $\bar{B}$ *is compact, then*

$$\frac{d}{dt}\Big|_0 \sigma(e^{tX}(B)) = \int_B \operatorname{div}_\sigma(X)\, d\sigma. \tag{7.2}$$

So $\operatorname{div}_\sigma(X)$ *measures the rate of spreading of the flow* e^{tX} *as seen by the measure* σ.

2. *Viewing* X *as a first order differential operator, for* $f \in C^1(M)$, *let*

$$X^*f = -Xf - \operatorname{div}_\sigma(X)\, f. \tag{7.3}$$

Then for all $f, g \in C^1(M)$ *such that the product* fg *has compact support,*

$$\int_M (Xf)\, g\, d\sigma = \int_M f(X^*g)\, d\sigma. \tag{7.4}$$

3. *The Radon-Nikodym derivative* Z_t *may be recovered from the flow* e^{tX} *and* $\operatorname{div}_\sigma(X)$ *by the formula*

$$Z_t(m) = e^{-\int_0^t \operatorname{div}_\sigma(X) \circ e^{-\tau X}(m)\, d\tau} \qquad (\forall m \in M). \tag{7.5}$$

Proof. The key point is that for $f: M \to \mathbb{R}$, bounded and measurable with compact support, we have by the definition of Z_t that

$$\int_M f \circ e^{tX}\, d\sigma = \int_M f Z_t\, d\sigma.$$

Hence

$$\frac{d}{dt}\int_M f\circ e^{tX}\,d\sigma=\int_M f\,\frac{\partial Z_t}{\partial t}\,d\sigma. \tag{7.6}$$

Taking f to be the characteristic function of B and $t=0$ in Eq. (7.6) implies Eq. (7.2). If we replace f by fg, where $f, g\in C^1(M)$ such that fg has compact support, then differentiating under the integral sign in Eq. (7.6) implies

$$\int_M \{X(f\circ e^{tX})\cdot g\circ e^{tX}+f\circ e^{tX}X(g\circ e^{tX})\}\,d\sigma=\int_M fg\,\frac{\partial Z_t}{\partial t}\,d\sigma. \tag{7.7}$$

Taking $t=0$ implies Eq. (7.3). Now suppose that $g\equiv 1$ in Eq. (7.7), then

$$\begin{aligned}\int_M f\,\frac{\partial Z_t}{\partial t}\,d\sigma&=\int_M X(f\circ e^{tX})\cdot 1\,d\sigma=\int_M (f\circ e^{tX})\cdot X^*1\,d\sigma\\&=-\int_M (f\circ e^{tX})\cdot \operatorname{div}_\sigma X\,d\sigma\\&=-\int_M (f\circ e^{tX})\cdot(\operatorname{div}_\sigma X)\circ e^{-tX}\circ e^{tX}\,d\sigma\\&=-\int_M f\cdot(\operatorname{div}_\sigma X)\circ e^{-tX}Z_t\,d\sigma.\end{aligned}$$

Since $f\in C_c^1(M)$ is arbitrary in this last equation, Z_t must satisfy the differential equation:

$$\frac{\partial Z_t}{\partial t}=-Z_t\cdot(\operatorname{div}_\sigma X)\circ e^{-tX}\qquad\text{with}\quad Z_0\equiv 1.$$

The unique solution of this equation is given in Eq. (7.5). Q.E.D.

For the infinite dimensional application to the loop group, it will be necessary to recall the following key estimate of Ana Bela Cruzeiro (see Corollary 2.2 in [2]) for the L^p-norms of Z_t in terms of $\operatorname{div}_\sigma X$. For the readers convenience I will also give the short proof.

THEOREM 7.3 (Cruzeiro). *Let $p\in(1,\infty)$ and M, X, σ, Z_t and $\alpha\equiv -\operatorname{div}_\sigma X$ be as above. Assume now that σ is probability measure and write $E_\sigma f$ for $\int_M f\,d\sigma$. If for a given $T>0$, $I(T)\equiv\sup_{|\tau|\leqslant T}E_\sigma[Z_\tau^p]<\infty$, then for all $t\in[-T,T]$;*

$$E_\sigma[Z_t^p]\leqslant\sup_{|s|\leqslant|t|}E_\sigma\exp\left\{\frac{p^2}{p-1}\,s\alpha\right\}\leqslant E_\sigma\exp\left\{\frac{p^2}{p-1}\,|t|\,|\alpha|\right\}. \tag{7.8}$$

Proof. Let $J_s \equiv [0, s]$ if $s \geqslant 0$ and $J_s \equiv [s, 0]$ if $s < 0$. Using Jensen's inequality, Eq. (7.5), the definition of Z_s and Holder's inequality we find for $|s| \leqslant |t| \leqslant T$ that

$$E_\sigma[Z_s^p] = E_\sigma \exp\left(p \int_0^s \alpha \circ e^{-\tau X}\, d\tau\right) = E_\sigma \exp\left(ps \int_{J_s} \alpha \circ e^{-\tau X} \frac{d\tau}{|s|}\right)$$

$$\leqslant E_\sigma \int_{J_s} \frac{d\tau}{|s|} \exp(ps\alpha \circ e^{-\tau X}) = \int_{J_s} \frac{d\tau}{|s|} E_\sigma(e^{ps\alpha} \cdot Z_{-\tau})$$

$$\leqslant \int_{J_s} \frac{d\tau}{|s|} (E_\sigma e^{pqs\alpha})^{1/q} (EZ_{-\tau}^p)^{1/p} \leqslant (E_\sigma e^{pqs\alpha})^{1/q} I(t)^{1/p},$$

where $1/q + 1/p = 1$. Hence it follows that

$$I(t) \leqslant \sup_{|s| \leqslant |t|} (E_\sigma e^{pqs\alpha})^{1/q} I(t)^{1/p}.$$

Solving this equation for $I(t)$ shows that

$$I(t) \leqslant \sup_{|s| \leqslant |t|} (E_\sigma e^{pqs\alpha}) = \sup_{|s| \leqslant |t|} E_\sigma \exp\left\{\frac{p^2}{p-1} s\alpha\right\} \leqslant E_\sigma \exp\left\{\frac{p^2}{p-1} |t|\, |\alpha|\right\}.$$

Q.E.D.

7.2. *Quasi-Invariance for the Heat Kernel Measure on $\mathscr{L}(G)$*

Let Σ be the $\mathscr{L}(G)$-valued Brownian motion constructed in Theorem 3.8 with $\Sigma(0, s) \equiv e$ for $0 \leqslant s \leqslant 1$ and $\nu_T \equiv \mathrm{Law}(\Sigma_T)$. For $h \in H_0(\mathfrak{g})$, let $\alpha_h \colon \mathscr{L}(G) \to \mathbb{R}$ be a Borel measurable function as in Corollary 6.6. By taking $u = f \in \mathscr{F}C^\infty(\mathscr{L})$ and $v = 1$ in Corollary 6.6 we find

$$E_{\nu_T}(\tilde{h}f) = E[(\tilde{h}f)(\Sigma_T)] = E[f(\Sigma_T)\, \alpha_h(\Sigma_T)] = E_{\nu_T}(f\alpha_h). \tag{7.9}$$

Since the flow of the vector field $\tilde{h}$ is $e^{t\tilde{h}}(g) = ge^{th}$, Eq. (7.9) and the finite dimensional discussion above motivates the following theorem.

THEOREM 7.4. *Let $h \in H_0(\mathfrak{g})$, then ν_T is quasi-invariant under the transformation $g \in \mathscr{L}(G) \to ge^h \in \mathscr{L}(G)$. Moreover, let Ω_0 denote the set of loops $g \in \mathscr{L}(G)$ such that $\int_0^1 |\alpha_h(ge^{-uh})|\, du < \infty$, then $\nu_T(\Omega_0) = 1$, the function $Z_h \colon \mathscr{L}(G) \to \mathbb{R}$ defined by*

$$Z_h(g) = 1_{\Omega_0}(g) \exp\left(\int_0^1 \alpha_h(ge^{-uh})\, du\right) \tag{7.10}$$

is in $L^1(d\nu_T)$ *and*

$$\int_{\mathscr{L}(G)} f(ge^h)\,d\nu_T(g)=\int_{\mathscr{L}(G)} f(g)\,Z_h(g)\,d\nu_T(g)$$

for all bounded measurable functions f *on* $\mathscr{L}(G)$. (*This last equation may also be equivalently expressed as* $E[f(\Sigma_T e^h)]=E[f(\Sigma_T)\,Z_h]$.)

Remark 7.5. For each finite partition of $\mathscr{P}$ of $[0,1]$, $\pi_{\mathscr{P}_*}\nu_T$ is the smooth measure on $G^{\mathscr{P}}$ given by $(d\pi_{\mathscr{P}_*\,T}/d\lambda_{\mathscr{P}})(x)=p_T^{\mathscr{P}}(e,x)$, where $x\in G^{\mathscr{P}}$, $p_T^{\mathscr{G}}$ is the heat kernel on $G^{\mathscr{P}}$ associated Riemannian inner product $(\cdot,\cdot)_{\mathscr{P}}$, i.e., $p_T^{\mathscr{P}}$ is the integral kernel of the operator $e^{T\Delta_{\mathscr{P}}/2}$.

Proof of Theorem 7.4. Let $\mathscr{P}_0\subset\mathscr{P}_1\subset\mathscr{P}_2\cdots$ be a nested sequence of partitions of $[0,1]$ such that $\lim_{n\to\infty}|\mathscr{P}_n|=0$. Suppose $n_0\in\{0,1,2,...\}$ and $f\colon\mathscr{L}\to\mathbb{R}$ is function such that $f=F\circ\pi_{\mathscr{P}_{n_0}}$ for some bounded Borel measurable function $F\colon G^{\mathscr{P}_{n_0}}\to\mathbb{R}$. Let $\mathscr{G}_n$ be the smallest σ-algebra on $\mathscr{L}(G)$ such that the projection $\pi_{\mathscr{P}_n}\colon\mathscr{L}(G)\to G^{\mathscr{P}_n}$ is measurable, where $G^{\mathscr{P}_n}$ is given the Borel σ-algebra. Set

$$\tilde{\alpha}_{h,n}\equiv-\operatorname{div}_{\pi_{\mathscr{P}_n}*\,\nu_T}\widetilde{h}\,|_{G^{\mathscr{P}_n}}$$

and $\alpha_{h,n}\equiv\tilde{\alpha}_{h,n}\circ\pi_{\mathscr{P}_n}$. Then by Proposition 7.2 and Corollary 6.6,

$$\alpha_{h,n}(\Sigma_T)\equiv\frac{1}{T}E\left[\int_0^T\left(\left\{I-\frac{1}{2}\tau\operatorname{Ric}\right\}H(\tau),\,\overleftarrow{d\beta}(\tau)\right)\,\middle|\,\mathscr{G}_n\right]\ \nu\text{-a.e.}\qquad(7.11)$$

Therefore, by Proposition 7.2 and Remark 7.5,

$$E[f(\Sigma_T e^h)]=E[f(\Sigma_T)\,Z_{h,n}],\qquad(7.12)$$

where

$$Z_{h,n}\equiv\exp\left\{\int_0^1\alpha_{h,n}(\Sigma_T e^{-sh})\,ds\right\}.\qquad(7.13)$$

The proof of Theorem 7.4 will continue after the following key lemma.

LEMMA 7.6. *Let* $Z_{h,n}$ *be defined as in Eq.* (7.13). *Then for all* $p\in(1,\infty)$,

$$\sup_n E_{\nu_T}Z_{h,n}^p\leqslant\exp\frac{1}{2}\left(\frac{p^4\,\|h\|^2}{(p-1)^2\,T^2}\int_0^T\left\|I-\frac{1}{2}\tau\operatorname{Ric}\right\|_{op}^2 d\tau\right)$$
$$=:M(p,h)<\infty,\qquad(7.14)$$

and $\{Z_{h,n}\}_{n=1}^{\infty}$ *is Cauchy in* $L^p(\nu_T)$.

Proof. Since $G^{\mathscr{P}_n}$ is compact and $g \in G^{\mathscr{P}_n} \to \exp \int_0^1 \tilde{\alpha}_{h,n}(ge^{-sh})\, ds$ is smooth, Cruzeiro's Theorem 7.3 may be applied to show

$$
\begin{aligned}
&E_{\nu_T} Z_{h,n}^p \\
&\quad \leqslant \sup_{|s| \leqslant 1} E_{\nu_T} \left[\exp \left\{ \frac{p^2}{p-1} s\alpha_{h,n} \right\} \right] \\
&\quad = \sup_{|s| \leqslant 1} E_{\nu_T} \left(\exp \left\{ \frac{p^2}{p-1} \frac{s}{T} E_{\nu_T} \left[\int_0^T \left(\left\{ I - \frac{1}{2} \tau \operatorname{Ric} \right\} H(\tau), \overleftarrow{d\beta}(\tau) \right) \middle| \mathscr{G}_n \right] \right\} \right) \\
&\quad \leqslant \sup_{|s| \leqslant 1} E_{\nu_T} \left(E_{\nu_T} \left[\exp \left\{ \frac{p^2}{p-1} \frac{s}{T} \int_0^T \left(\left\{ I - \frac{1}{2} \tau \operatorname{Ric} \right\} H(\tau), \overleftarrow{d\beta}(\tau) \right) \right\} \middle| \mathscr{G}_n \right] \right) \\
&\quad = \sup_{|s| \leqslant 1} E_{\nu_T} \left[\exp \left\{ \frac{p^2}{p-1} \frac{s}{T} \int_0^T \left(\left\{ I - \frac{1}{2} \tau \operatorname{Ric} \right\} H(\tau), \overleftarrow{d\beta}(\tau) \right) \right\} \right],
\end{aligned} \tag{7.15}
$$

wherein the second inequality we use Jensen's inequality. For fixed $s \in [-1, 1]$ and T as above, set

$$
M_t \equiv \frac{p^2}{p-1} \frac{s}{T} \int_{T-t}^T \left(\left\{ I - \frac{1}{2} \tau \operatorname{Ric} \right\} H(\tau), \overleftarrow{d\beta}(\tau) \right) \qquad \text{for} \quad 0 \leqslant t \leqslant T.
$$

Then M_t is a martingale such that the quadratic variation of M at T is given by

$$
\langle M \rangle_T = \frac{p^4 s^2}{(p-1)^2\, T^2} \int_0^T \left\| \left\{ I - \frac{1}{2} \tau \operatorname{Ric} \right\} H(\tau) \right\|^2 d\tau.
$$

and thus, because $\|H(\tau)\| = \|h\|$,

$$
\|\langle M \rangle_T\|_{L^\infty(\nu)} \leqslant \frac{p^4\, \|h\|^2\, s^2}{(p-1)^2\, T^2} \int_0^T \left\| I - \frac{1}{2} \tau \operatorname{Ric} \right\|_{op}^2 d\tau < \infty. \tag{7.16}
$$

Hence, Novikov's criterion (see Proposition 1.15, p. 308 in [24]) implies that $Ee^{M_T - (1/2)\langle M \rangle_T} = 1$ so that

$$
\begin{aligned}
Ee^{M_T} &= E(e^{M_T - (1/2)\langle M \rangle_T} \cdot e^{(1/2)\langle M \rangle_T}) \leqslant e^{(1/2)\, \|\langle M \rangle_T\|_{L^\infty}} \\
&\leqslant \exp \frac{1}{2} \left(\frac{s^2 p^4\, \|h\|^2}{(p-1)^2\, T^2} \int_0^T \left\| I - \frac{1}{2} \tau \operatorname{Ric} \right\|_{op}^2 d\tau \right).
\end{aligned} \tag{7.17}
$$

(Alternatively, see Lemma 1.4 in Kusuoka and Stroock [16].) Combining equations (7.15) and (7.17) proves the bound in Eq. (7.14).

Let $m > n$ be two positive integers. By the fundamental theorem of calculus, for all $x, y \in \mathbb{R}$,

$$e^y - e^x = (y-x) \int_0^1 e^{(uy+(1-u)x)}\, du = (y-x) \int_0^1 (e^y)^u (e^x)^{1-u}\, du.$$

Applying this equation with $x = \int_0^1 \alpha_{h,n}(\Sigma_T e^{-sh})\, ds$ and $y = \int_0^1 \alpha_{h,m}(\Sigma_T e^{-sh})\, ds$ gives

$$|Z_{h,n} - Z_{h,m}| = \left| \int_0^1 \alpha_{h,n}(\Sigma_T e^{-sh}) - \alpha_{h,m}(\Sigma_T e^{-sh})\, ds \right| \cdot \int_0^1 Z_{h,n}^u Z_{h,m}^{(1-u)}\, du.$$

Using Holder's inequality we find

$$E\,|Z_{h,n} - Z_{h,m}| \leqslant \int_0^1 \|\alpha_{h,n}(\Sigma_T e^{-sh}) - \alpha_{h,m}(\Sigma_T e^{-sh})\|_{L^{3/2}}\, ds \int_0^1 \|Z_{h,n}^u Z_{h,m}^{(1-u)}\|_{L^3}\, du. \tag{7.18}$$

Now by the bound in Eq. (7.14)

$$\|Z_{h,n}^u Z_{h,m}^{(1-u)}\|_{L^3} \leqslant \|Z_{h,n}^u\|_{L^6}\, \|Z_{h,m}^{(1-u)}\|_{L^6} \leqslant M(6,h)^{1/3},$$

which combined with Eq. (7.18) shows that

$$E\,|Z_{h,n} - Z_{h,m}| \leqslant M(6,h)^{1/3} \int_0^1 \|\alpha_{h,n}(\Sigma_T e^{-sh}) - \alpha_{h,m}(\Sigma_T e^{-sh})\|_{L^{3/2}}\, ds. \tag{7.19}$$

Since $m > n$,

$$\begin{aligned}
&\|\alpha_{h,n}(\Sigma_T e^{-sh}) - \alpha_{h,m}(\Sigma_T e^{-sh})\|_{L^{3/2}} \\
&\quad = [E(|\alpha_{h,n}(\Sigma_T) - \alpha_{h,m}(\Sigma_T)|^{3/2}\, Z_{-sh,m})]^{2/3} \\
&\quad = \|\,|\alpha_{h,n}(\Sigma_T) - \alpha_{h,m}(\Sigma_T)|\, (Z_{-sh,m})^{2/3}\|_{L^{3/2}} \\
&\quad \leqslant \|\alpha_{h,n}(\Sigma_T) - \alpha_{h,m}(\Sigma_T)\|_{L^2} \cdot \|(Z_{-sh,m})^{2/3}\|_{L^6} \\
&\quad \leqslant (M(4,h))^{1/6} \cdot \|\alpha_{h,n}(\Sigma_T) - \alpha_{h,m}(\Sigma_T)\|_{L^2}.
\end{aligned}$$

This equation and Eq. (7.19) shows that

$$E\,|Z_{h,n}-Z_{h,m}| \leqslant M(6,h)^{1/3}\,(M(4,h))^{1/6}\cdot \|\alpha_{h,n}(\Sigma_T)-\alpha_{h,m}(\Sigma_T)\|_{L^2(\nu)}. \tag{7.20}$$

Now

$$\|\alpha_{h,n}(\Sigma_T)-\alpha_{h,m}(\Sigma_T)\|_{L^2} \equiv \frac{1}{T}\left\|(E_n-E_m)\int_0^T\left(\left\{I-\frac{1}{2}\tau\,\mathrm{Ric}\right\}H(\tau),\,\overleftarrow{d\beta}(\tau)\right)\right\|_{L^2}, \tag{7.21}$$

where $E_n \equiv E[\,\cdot \mid \mathscr{G}_n]$ denotes conditional expectation relative to the σ-algebra $\mathscr{G}_n$. Since E_n converges strongly to $E[\,\cdot \mid \sigma(\Sigma_T)]$ in L^2 it follows from Eq. (7.21) that

$$\lim_{m,n\to\infty} \|\alpha_{h,n}(\Sigma_T)-\alpha_{h,m}(\Sigma_T)\|_{L^2}=0.$$

In view of Eq. (7.20) this finishes the proof of the lemma. Q.E.D.

We now continue the proof of Theorem 7.4. Let $\tilde{Z}_h$ be the $L^p(\nu_T)$-limit of $Z_{h,n}$. Of course $\tilde{Z}_h$ inherits the bounds in Eq. (7.14), namely that

$$E_{\nu_T}\tilde{Z}_h^p \leqslant M(p,h)<\infty \qquad \text{for all} \quad p\in(1,\infty).$$

By the previous lemma, we may let n tend to infinity in Eq. (7.12) to find

$$E[f(\Sigma_T e^h)] = E[f(\Sigma_T)\,\tilde{Z}_h(\Sigma_T)]. \tag{7.22}$$

Eq. (7.22) is valid for all $f\in \mathscr{F}C^\infty(\mathscr{L})$ which are based on $\mathscr{P}_n$ for some positive integer n. So by a monotone class argument or Dynkin's $\pi-\lambda$ theorem, one may easily show that this equation is in fact valid for all bounded measurable functions on $\mathscr{L}(G)$.

Setting $\sigma\equiv\nu_T$, and $\sigma_h\equiv R_{e^h*}\nu_T$, we have shown that $\sigma_h \ll \sigma$ and that $d\sigma_h/d\sigma=\tilde{Z}_h$. We now show that $\sigma\ll\sigma_h$. To this end let $f\colon \mathscr{L}(G)\to\mathbb{R}$ be a bounded measurable function, then

$$\begin{aligned}E_\sigma(f) &= Ef(\Sigma_T)=Ef(\Sigma_T e^{-h}e^h)\\ &= E[f(\Sigma_T e^h)\,\tilde{Z}_{-h}(\Sigma_T)] = E[f(\Sigma_T e^h)\,\tilde{Z}_{-h}(\Sigma_T e^h e^{-h})]\\ &= E_{\sigma_h}(f\tilde{Z}_{-h}((\cdot)\,e^{-h})).\end{aligned}$$

Therefore $\sigma\ll\sigma_h$ and $d\sigma/d\sigma_h=\tilde{Z}_{-h}((\cdot)\,e^{-h})$.

So to finish the proof we need only show that Z_h defined in Eq. (7.10) is well defined and $Z_h(\Sigma_T) = \tilde{Z}_h$ ν-a.e. First consider

$$\int_{\mathscr{L}(G)} \left(\int_0^1 |\alpha_h(ge^{-uh})| \, du \right)^2 d\nu_T(g) = E \left(\int_0^1 |\alpha_h(\Sigma_T e^{-uh})| \, du \right)^2$$

$$\leqslant \int_0^1 E \, |\alpha_h(\Sigma_T e^{-uh})|^2 \, du$$

$$= \int_0^1 E(|\alpha_h(\Sigma_T)|^2 \, \tilde{Z}_{-uh}(\Sigma_T)) \, du$$

$$\leqslant (E \, |\alpha_h(\Sigma_T)|^4)^{1/2} \, M(2, h) < \infty.$$

(Note that $E \, |\alpha_h(\Sigma_T)|^4 < \infty$ because of Remark 6.3 and the fact that conditional expectations are contractions on L^p-spaces.) This shows that $\int_0^1 |\alpha_h(ge^{-uh})| \, du < \infty$ for ν_T-a.e. g and hence that Z_h is well defined.

Set $\xi \equiv \int_0^1 \alpha_h(\Sigma_T e^{-uh}) \, du$ and $\xi_n \equiv \int_0^1 \alpha_{h,n}(\Sigma_T e^{-uh}) \, du$. Since $Z_h(\Sigma_T) = e^{\xi}$ and $\tilde{Z}_h(\Sigma_T) = \lim_{n \to \infty} e^{\xi_n}$, to show that $Z_h(\Sigma_T) = \tilde{Z}_h(\Sigma_T)$ a.e. it suffices to show that ξ_n converges to ξ in L^1. We start with the estimate

$$E \, |\xi - \xi_n| \leqslant \int_0^1 E \, |\alpha_{h,n}(\Sigma_T e^{-uh}) - \alpha_h(\Sigma_T e^{-uh})| \, du$$

$$= \int_0^1 E[\, |\alpha_{h,n}(\Sigma_T) - \alpha_h(\Sigma_T)| \, \tilde{Z}_{-uh}(\Sigma_T)] \, du$$

$$\leqslant \|\alpha_{h,n}(\Sigma_T) - \alpha_h(\Sigma_T)\|_{L^2} \int_0^1 \|\tilde{Z}_{-uh}(\Sigma_T)\|_{L^2(P)} \, du$$

$$\leqslant \|\alpha_{h,n}(\Sigma_T) - \alpha_h(\Sigma_T)\|_{L^2(P)} \sqrt{M(2, h)}.$$

Now by Eq. (7.11),

$$\|\alpha_{h,n}(\Sigma_T) - \alpha_h(\Sigma_T)\|_{L^2(P)}$$

$$\equiv \frac{1}{T} \left\| (E_n - E_\infty) \int_0^T \left(\left\{ I - \frac{1}{2} \tau \, \mathrm{Ric} \right\} H(\tau), \overleftarrow{d\beta}(\tau) \right) \right\|_{L^2(P)}, \tag{7.23}$$

where $E_n \equiv E[\,\cdot \mid \mathscr{G}_n]$ and $E_\infty(\cdot) \equiv E[\,\cdot \mid \sigma(\Sigma_T)]$. This finishes the proof because E_n converges strongly in L^2 to E_∞ as $n \to \infty$. Q.E.D.

We now wish to extend Theorem 7.4 to include right translations by $k \in \mathscr{L}_0^1(G)$, where $\mathscr{L}_0^1(G)$ denotes the space of contractible loops in $\mathscr{L}(G)$

which have finite energy. A loop $k \in \mathscr{L}(G)$ is said to have *finite energy* provided k is absolutely continuous and

$$\int_0^1 |\theta\langle k'(s)\rangle|^2 \, ds = \int_0^1 |L_{k(s)^{-1}*} k'(s)|^2 \, ds < \infty.$$

COROLLARY 7.7. *For each $k \in \mathscr{L}_0^1(G)$, ν_T quasi-invariant under the right translation map $R_k \colon \mathscr{L}(G) \to \mathscr{L}(G)$ given by $R_k g = gk$.*

We will need the following simple lemma for the proof of this corollary.

LEMMA 7.8. *Let $(\mathscr{L}, \mathscr{G}, \mu)$ be a probability space and $T \colon \mathscr{L} \to \mathscr{L}$ be an invertible measurable map with a measurable inverse. Assume that $T_* \mu \equiv \mu \circ T^{-1}$, $T_*^{-1} \mu \equiv \mu \circ T$, and μ are mutually absolutely continuous. Also let $f \colon \mathscr{L} \to [0, \infty)$ be a measurable function such that $f > 0$ μ-a.s. and $\int_{\mathscr{L}} f \, du = 1$. If $f\mu$ denotes the probability measure defined by $(f\mu)(A) \equiv \int_A f \, d\mu$ for all $A \in \mathscr{G}$, then $T_*(f\mu) \equiv (f\mu) \circ T^{-1}$, $T_*^{-1}(f\mu) \equiv (f\mu) \circ T$, and μ are all mutually absolutely continuous as well.*

Proof. Let $Z = dT_* \mu / d\mu$ and $g \colon \mathscr{L} \to [0, \infty)$ be a measurable function. Then

$$\int_{\mathscr{L}} g \, d[T_*(f\mu))] = \int_{\mathscr{L}} g \circ T \cdot f \, d\mu = \int_{\mathscr{L}} (g \cdot f \circ T^{-1}) \circ T \, d\mu$$

$$= \int_L g \cdot (f \circ T^{-1}) \, Z \, d\mu.$$

This shows that

$$\frac{d[T_*(f\mu))]}{d\mu} = Z \cdot (f \circ T^{-1}). \tag{7.24}$$

Since $\mu(f^{-1}(\{0\})) = 0$ and μ and $T_*^{-1}\mu$ are mutually absolutely continuous, we have $0 = (\mu \circ T)(f^{-1}(\{0\})) = \mu(\{g \in \mathscr{L} : f \circ T^{-1}(g) = 0\})$. That is $f \circ T^{-1}$ is positive μ-a.s. Thus it follows from Eq. (7.24) that $T_*(f\mu)$ and μ are mutually absolutely continuous. By symmetry, $T_*^{-1}(f\mu)$ and μ are mutually absolutely continuous as well. Q.E.D.

Proof of Corollary 7.7. By Theorem 7.4 and repeated use of Lemma 7.8, it suffices to prove: for any $k \in \mathscr{L}_0^1(G)$ there is an integer n and $h_i \in H_0(\mathfrak{g})$ such that

$$k(s) = e^{h_1(s)} e^{h_2(s)} \cdots e^{h_{n+1}(s)}. \tag{7.25}$$

To prove (7.25), choose a ball B in $\mathfrak{g}$ centered at 0 such that $V:=\{e^{\xi}: \xi \in B\}$ is open and the map $\xi \in B \to e^{\xi} \in V$ is a diffeomorphism with inverse denoted by log. Let $\mathscr{L}(V)=\{g \in \mathscr{L}(G): g([0,1]) \subset V\}$, so that $\mathscr{L}(V)$ is an open neighborhood of $\mathscr{L}_0(G)$. It is easily shown that $W \equiv \bigcup_{n=1}^{\infty} \mathscr{L}(V)^n$ is both open and closed in $\mathscr{L}(G)$ (with the sup-norm topology) and hence $W=\mathscr{L}_0(G)$-the connected component of the identity in $\mathscr{L}(G)$. (The space $\mathscr{L}_0(G)$ may also be described as the space of contractible loops in $\mathscr{L}(G)$.) Therefore there is an integer $n \in \mathbb{Z}_+$ and $k_i \in \mathscr{L}(V)$ such that $k=k_1 k_2 \cdots k_n$. Let $u_i(s) \equiv \log k_i(s)$, then $u_i \in \mathscr{L}(\mathfrak{g})$ and $e^{-u_n} e^{-u_{n-1}} \cdots e^{-u_1} k$ is the constant path sitting at $e \in G$. Choose $h_i \in H_0(\mathfrak{g})$ sufficiently close to u_i in the sup-norm topology on $\mathscr{L}(\mathfrak{g})$ such that $e^{-h_n} e^{-h_{n-1}} \cdots e^{-h_1} k \in \mathscr{L}(V)$, Define $h_{n+1} \equiv \log(e^{-h_n} e^{-h_{n-1}} \cdots e^{-h_1} k) \in H_0(\mathfrak{g})$. Then

$$e^{h_{n+1}}=e^{-h_n} e^{-h_{n-1}} \cdots e^{-h_1} k,$$

which is equivalent to (7.25). Q.E.D.

Proposition 7.9. *The heat kernel measure ν_T is invariant relative to the inverse map $g \in \mathscr{L}(G) \to g^{-1} \in \mathscr{L}(G)$.*

Proof. It suffices to show that each of the finite dimensional distributions, $\pi_{\mathscr{P}*} \nu_T \equiv \nu_T \circ \pi_{\mathscr{P}}^{-1}$ (where $\mathscr{P}$ is a finite partition of $[0,1]$) is invariant under the inverse map $g \in G^{\mathscr{P}} \to g^{-1} \in G^{\mathscr{P}}$. But this property is know to hold, in general, for heat kernel measures on uni-modular Lie groups equipped with a left invariant Riemannian metric, see for example Remark 2.2 and Proposition 3.1 in Driver and Gross [7] Q.E.D.

Corllary 7.10. *For each $k \in \mathscr{L}_0^1(G)$, ν_T is quasi-invariant under the left translation map $L_k: \mathscr{L}(G) \to \mathscr{L}(G)$ given by $L_k g=kg$.*

Proof. This a direct consequence of Corollary 7.7 and Proposition 7.9 above. Indeed, let $f: \mathscr{L} \to \mathbb{R}$ be a bounded and measurable function and for each $k \in \mathscr{L}_0^1(G)$ set $Z_k \equiv d\nu_T \circ R_k^{-1}/d\nu_T$. Then

$$\begin{aligned} Ef(k\Sigma_T) &= Ef(k\Sigma_T^{-1})=Ef((\Sigma_T k^{-1})^{-1}) \\ &= E(f(\Sigma_T^{-1})\, Z_{k^{-1}}(\Sigma_T))=E(f(\Sigma_T)\, Z_{k^{-1}}(\Sigma_T^{-1})). \end{aligned}$$

This shows that $\nu_T \circ L_k^{-1} \ll \nu_T$ and $d\nu_T \circ L_k^{-1}/d\nu_T(g)=Z_{k^{-1}}(g^{-1})$ for ν_T almost every $g \in \mathscr{L}(G)$. Since $Z_{k^{-1}}>0$ ν_T-a.s. and $g \to Z_{k^{-1}}(g^{-1})$ has the same distribution as $Z_{k^{-1}}$, it follows that $d\nu_T \circ L_k^{-1}/d\nu_T>0$ ν_T-a.s. Hence ν_T is absolutely continuous relative to $\nu_T \circ L_k^{-1}$ as well. Q.E.D.

8. APPENDIX: REVIEW OF THE ITÔ INTEGRAL IN INFINITE DIMENSIONS

As in the body of the text, let $(\mathscr{W}, \mathscr{F}, \{\mathscr{F}_t\}_{t \geqslant 0}, P)$ be a filtered probability space satisfying the usual hypothesis as described in the beginning of Section 3.1. The purpose of this appendix is to set up notation and review some very basic facts about Hilbert space valued martingales and the Itô integral $\int F\, d\beta$. For Hilbert space martingale theory the reader is referred to Métivier [18]. For the Itô integral on abstract Wiener space, see Sections III.5 of Kuo [15], p. 188–207, especially Theorem 5.1 of [15]. Also see Kusuoka and Stroock [16] p. 5 for a very short description of the Itô integral in this context. For the notion and basic properties of conditional expectations for Banach space valued Random variables, see Section 8.3 in Chapter 2 in Métivier [18].

8.1. *Continuous Hilbert Valued Local Martingales*

Let K be a Hilbert space. We will use $(\cdot, \cdot)$ to denote the inner product on both of the Hilbert space $H_0(\mathfrak{g})$ and K.

THEOREM 8.1 (Quadratic Variations). *Suppose that M and N are two continuous local martingales with values in a Hilbert space K. Then there is a real valued process of bounded variation $\prec M, N \succ$ such that for any increasing sequence of partitions $\{\pi_n\}_{n=1}^{\infty}$ of $[0, \infty)$ such that $|\pi_n| \to 0$ as $n \to \infty$,*

$$\prec M, N \succ_t = \lim_{n \to \infty} \sum_{t_i \in \pi_n}^{\infty} (M_{t \wedge t_{i+1}} - M_{t \wedge t_i}, N_{t \wedge t_{i+1}} - N_{t \wedge t_i}), \tag{8.1}$$

where the limit exists in probability uniformly for t in compact subsets of $[0, \infty)$. Moreover the following properties hold:

1. $|\prec M, N \succ| \leqslant \sqrt{\prec M \succ \cdot \prec N \succ}$ *a.s., where* $\prec M \succ \equiv \prec M, M \succ$.
2. $|\prec M \succ - \prec N \succ| \leqslant \sqrt{\prec M - N \succ \cdot \prec M + N \succ}$ *a.s.*
3. $EM_t^{*2} \leqslant 4E\, \|M_0\|^2 + 4E\prec M \succ_t$.
4. *The following three conditions are equivalent*:

   (a) *M is a square integrable martingale,*

   (b) *$EM_t^{*2} < \infty$ for all $t \geqslant 0$ and*

   (c) *$E\, \|M_0\|^2 < \infty$ and $E\prec M \succ(t) < \infty$ for all $t \geqslant 0$.*

5. *If* M_n *and* M *are continuous* K*-valued* L^2*-martingales such that* $E\,\|(M_n - M)_t\|^2 \to 0$ *as* $n \to \infty$ *then* $\langle M_n \rangle(t) \to \langle M \rangle(t)$ *in* L^1.

6. *Suppose the* $\{M_n\}_{n=1}^{\infty}$ *is a sequence of* K*-valued continuous local martingales such that* $M_n(0) = 0$ *for all* n *and* $\langle M_n \rangle \to 0$ *a.s. as* $n \to \infty$. *Then* $M_n \to 0$ *in probability uniformly on compact subsets of* $[0, \infty)$.

For a proof of this theorem see, for example, Theorems 20.5 and 20.6 in Métivier [18] and Métivier and Pellaumail [19].

8.2. *The Itô Integral on Our Abstract Wiener Space*

For the rest of this Appendix we will adopt the notation in Section 3.1 of the body of the paper.

THEOREM 8.2. *Suppose that* $\{f_t\}_{t \geqslant 0}$ *is an* $(\mathcal{W}, \{\mathcal{F}_t\}_{t \geqslant 0}, \mathcal{F}, P)$*-adapted and continuous process with values in* $H_0(\mathfrak{g})$. *Then there is a continuous local martingale* N *such that for any orthonormal basis* $\{h_n\}_{n=1}^{\infty}$ *of* $H_0(\mathfrak{g})$, $N = \sum_{n=1}^{\infty} \int (f_t, h_n)\, d\beta^{h_n}$, *where the sum is convergent in probability uniformly for* t *in compact subsets of* $[0, \infty)$. *We will write* N_t *as* $\int_0^t f\, d\beta$ *or* $N = \int f\, d\beta$ *for short. The quadratic variation of* $\int f\, d\beta$ *is given by* $\langle \int f\, d\beta \rangle_t = \int_0^t \|f_\tau\|^2\, d\tau$ *or* $\langle \int f\, d\beta \rangle = \int \|f\|^2\, d\tau$ *for short.*

Proof. Let $k \in \mathbb{Z}_+$ and $N^{(k)} \equiv \sum_{n=1}^{k} \int (f_t, h_n)\, d\beta^{h_n}$—a local martingale. Then for $k' > k$,

$$\langle N^{(k')} - N^{(k)} \rangle = \int \sum_{n=k+1}^{k'} |(f_t, h_n)|^2\, d\tau \to 0 \qquad \text{a.s. as} \quad k, k' \to \infty.$$

Using theorem 8.1, this shows that $N^{(k)}$ converges uniformly on compacts in probability to a local martingale N and moreover

$$\langle N \rangle = \int \sum_{n=1}^{\infty} |(f_\tau, h_n)|^2\, d\tau = \int \|f_\tau\|^2\, d\tau.$$

Now suppose that $\{l_m\}_{m=1}^{\infty}$ is another orthonormal basis for $H_0(\mathfrak{g})$ and that

$$Q^{(k)} \equiv \sum_{m=1}^{k} \int (f_t, l_m)\, d\beta^{l_m}.$$

Let P_k and $\tilde{P}_k$ be orthogonal projections onto $\operatorname{span}\{h_1, h_2, ..., h_k\}$ and $\operatorname{span}\{l_1, l_2, ..., l_k\}$ respectively. Then

$$\begin{aligned}
\prec Q^{(k)} - N^{(k)} \succ &= \prec \sum_{n=1}^{k} \left\{ \int (f_t, h_n)\, d\beta^{h_n} - \int (f_t, l_n)\, d\beta^{l_n} \right\} \succ \\
&= \sum_{n,\, m=1}^{k} \prec \left\{ \int (f_t, h_n)\, d\beta^{h_n} - \int (f_t, l_n)\, d\beta^{l_n} \right\}, \\
&\qquad \left\{ \int (f_t, h_m)\, d\beta^{h_m} - \int (f_t, l_m)\, d\beta^{l_m} \right\} \succ \\
&= \sum_{n}^{k} \int \{ |(f_t, h_n)|^2 + |(f_t, l_n)|^2 \}\, dt \\
&\quad - 2 \sum_{n,\, m=1}^{k} \int (f_t, h_m)(f_t, l_m)(h_n, l_m)\, dt \\
&= \int \{ \|P_k f_t\|^2 + \|\tilde{P}_k f_t\|^2 - 2(P_k f_t, \tilde{P}_k f_t) \}\, dt \to 0 \\
&\qquad \text{a.s. as} \quad k \to \infty,
\end{aligned}$$

where we have used the fact that P_k and $\tilde{P}_k$ are strongly convergent to I as $k \to \infty$ along with the dominated convergence theorem. This shows that $\int f\, d\beta$ is basis independent. Q.E.D.

THEOREM 8.3 (Associativity). *Suppose $\{f_t\}_{t \geqslant 0}$ and $\{g_t\}_{t \geqslant 0}$ are $(\mathscr{W}, \{\mathscr{F}_t\}_{t \geqslant 0}, \mathscr{F}, P)$-adapted and continuous process with values in $H_0(\mathfrak{g})$ and $\mathbb{R}$ respectively. Set $M \equiv \int (f, d\beta)$, then*

$$\int g\, dM = \int (gf, d\beta). \tag{8.2}$$

Proof. Let $\{h_n\}_{=1}^{\infty}$ be orthonormal basis for $H_0(\mathfrak{g})$ and for $N \in \mathbb{Z}_+$ set

$$M_N \equiv \sum_{n=1}^{N} \int (f, h_n)\, d\beta^{h_n}.$$

Then

$$\prec \int g\, dM - \int g\, dM_N \succ = \int g^2\, d\prec M - M_N \succ = \int g^2 \left(\sum_{n=N+1}^{\infty} |(f, h_n)|^2 \right) dt,$$

and this last expression tends to zero almost surely as $N \to \infty$. Therefore $\int g\, dM_N \to \int g\, dM$ uniformly on compacts in probability. On the other hand, by associativity of the finite dimensional Itô integral,

$$\int g\, dM_N = \sum_{n=1}^{N} \int g d\left(\int (f, h_n)\, d\beta^{h_n}\right) = \sum_{n=1}^{N} \int g(f, h_n)\, d\beta^{h_n}$$

$$= \sum_{n=1}^{N} \int (gf, h_n)\, d\beta^{h_n} \to \int (gf, d\beta) \qquad \text{as} \quad N \to \infty.$$

So we have shown that $\int g\, dM_N$ converges to both $\int g\, dM$ and $\int (gf, d\beta)$.

Q.E.D.

THEOREM 8.4. *Suppose that* K *is another separable Hilbert space,* $\{F_t\}_{t \geqslant 0}$ *is an* $\mathscr{F}_t$*-adapted and continuous process with values in* $HS(H_0(\mathfrak{g}), K)$*—the Hilbert Schmidt operators from* $H_0(\mathfrak{g})$ *to* K*. Then there is a continuous* K*-values local martingale* N *such that, for any orthonormal basis* $\{k_n\}_{n=1}^{\infty}$ *of* K*,* $\sum_{n=1}^{\infty} \int (F^* k_n, d\beta)\, k_n$ *converges uniformly on compacts in probability to* N*. We will write* N_t *as* $\int_0^t F\, d\beta$ *or* $N = \int F\, d\beta$ *for short. The quadratic variation of* $\int F\, d\beta$ *is given by* $\langle \int F\, d\beta \rangle_t = \int_0^t \|F\|_{HS}^2\, d\tau$ *or* $\langle \int F\, d\beta \rangle = \int \|F\|_{HS}^2\, d\tau$*.*

Proof. Let $k \in \mathbb{Z}_+$ and $N^{(k)} \equiv \sum_{n=1}^{k} \int (F^* k_n, d\beta)\, k_n$—a K-valued local martingale. Then for $k' > k$,

$$\langle N^{(k')} - N^{(k)} \rangle = \left\langle \int \sum_{n=k+1}^{k'} \int (F^* k_n, d\beta)\, k_n \right\rangle$$

$$= \sum_{m,\, n=k+1}^{k'} \left\langle \int (F^* k_n, d\beta), \int (F^* k_m, d\beta) \right\rangle (k_n, k_m)$$

$$= \sum_{n=k+1}^{k'} \int \|F^* k_n\|_{HS}^2\, d\tau.$$

Recall that

$$\|F\|_{HS}^2 = \sum_{n=1}^{\infty} \|F h_n\|^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |(F h_n, k_m)|^2$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |(h_n, F^* k_m)|^2 = \|F^*\|_{HS}^2.$$

Therefore, $\sum_{n=1}^{\infty} \int \|F^* k_n\|^2 \, d\tau < \infty$ and hence,

$$\prec N^{(k')} - N^{(k)} \succ = \sum_{n=k+1}^{k'} \int \|F^* k_n\|^2 \, d\tau \to 0 \qquad \text{as} \quad k, k' \to \infty.$$

Again by Theorem 8.1, this shows that $N^{(k)}$ converges uniformly on compacts in probability to a K-valued local martingale N and also that

$$\prec N \succ = \int \|F_\tau\|_{HS}^2 \, d\tau.$$

Now suppose that $\{l_m\}_{m=1}^{\infty}$ is another orthonormal basis for $H_0(\mathfrak{g})$ and that

$$Q^{(k)} \equiv \sum_{m=1}^{k} \int (F^* l_m, d\beta) \, l_m.$$

Again let P_k be orthogonal projection onto span $\{k_1, k_2, ..., k_k\}$. Then

$$\begin{aligned}
\prec Q^{(k)} - N^{(k)} \succ &= \prec \sum_{n=1}^{k} \left\{ \int (F^* k_n, d\beta) \, k_n - \int (F^* l_n, d\beta) \, l_n \right\} \succ \\
&= \prec \sum_{n=1}^{k} \int (F^* k_n, d\beta) \, k_n \succ + \prec \sum_{n=1}^{k} \int (F^* l_n, d\beta) \, l_n \succ \\
&\quad - 2 \sum_{n=1}^{k} \sum_{m=1}^{k} \prec \int (F^* k_n, d\beta) \, k_n, \int (F^* l_m, d\beta) \, l_m \succ \\
&= \sum_{n=1}^{k} \int \|F^* k_n\|^2 \, d\tau + \sum_{n=1}^{k} \int \|F^* l_n\|^2 \, d\tau \\
&\quad - 2 \int G_k \, d\tau, \qquad (8.3)
\end{aligned}$$

where

$$G_k \equiv \sum_{n=1}^{k} \sum_{m=1}^{k} (F^* k_n, F^* l_m)(k_n, l_m) = \sum_{m=1}^{k} (F^* P_k l_m, F^* l_m).$$

Let $P'_k \equiv I - P_k$ and notice that

$$\begin{aligned}
|G_k| &\leqslant \sqrt{\sum_{m=1}^{\infty} \|F^* P_k l_m\|^2} \sqrt{\sum_{m=1}^{\infty} \|F^* l_m\|^2} \\
&= \|F^* P_k\|_{HS} \cdot \|F^*\|_{HS} \leqslant \|F\|_{HS}^2
\end{aligned}$$

and

$$\begin{aligned}\left|G_k - \sum_{m=1}^{k} (F^*l_m, F^*l_m)\right| &\leqslant \sum_{m=1}^{k} |(F^*P'_k l_m, F^*l_m)| \\ &\leqslant \sqrt{\sum_{m=1}^{k} \|F^*P'_k l_m\|^2} \cdot \sqrt{\sum_{m=1}^{k} \|F^*l_m\|^2} \\ &\leqslant \|F^*P'_k\|_{HS} \cdot \|F^*\|_{HS} \\ &= \sqrt{\sum_{m=k+1}^{\infty} \|F^*h_m\|^2} \cdot \|F\|_{HS}.\end{aligned}$$

Therefore we are justified in applying the dominated convergence theorem in Eq. (8.3) to find that

$$\lim_{k\to\infty} \prec Q^{(k)} - N^{(k)} \succ = \int \{ \|F^*\|_{HS}^2 + \|F^*\|_{HS}^2 - 2\,\|F^*\|_{HS}^2\}\, dt = 0.$$

Hence $Q^{(k)} - N^{(k)} \to 0$ in probability which proves that $\int F\, d\beta$ is basis independent. Q.E.D.

THEOREM 8.5 (Itô's Lemma). *Suppose that K is a separable Hilbert space, $\{F_t\}_{t\geqslant 0}$ is an $(\mathscr{W}, \{\mathscr{F}_t\}_{t\geqslant 0}, \mathscr{F}, P)$-adapted and continuous process with values in $HS(H_0(\mathfrak{g}), K)$. Then*

$$\left\|\int_0^t F\, d\beta\right\|^2 = 2\int_0^t \left(F_\tau^* \int_0^\tau F\, d\beta, d\beta(\tau)\right) + \int_0^t \|F_\tau\|_{HS}^2\, d\tau. \tag{8.4}$$

Proof. Let $\{k_n\}_{n=1}^{\infty}$ be orthonormal basis for K and $M_n \equiv \int (F^*k_n, d\beta)$. Then

$$\begin{aligned}\left\|\int F\, d\beta\right\|^2 &= \sum_{n=1}^{\infty} \left(\int (F^*k_n, d\beta)\right)^2 = \sum_{n=1}^{\infty} M_n^2 \\ &= \sum_{n=1}^{\infty} \left\{2\int M_n\, dM_n + \prec M_n \succ\right\} \\ &= 2\sum_{n=1}^{\infty} \int (M_n F^*k_n, d\beta) + \sum_{n=1}^{\infty} \int \|F^*k_n\|^2\, d\tau \\ &= 2\sum_{n=1}^{\infty} \int (M_n F^*k_n, d\beta) + \int \|F\|_{HS}^2\, d\tau,\end{aligned} \tag{8.5}$$

wherein the third equality we used the Associativity Theorem 8.3.

Let P_N be orthogonal projection onto the subspace spanned by $\{k_1, k_2, ..., k_N\}$. Then for any $h \in H_0(\mathfrak{g})$

$$\begin{aligned}\left(h, F^* \int P_N F\, d\beta\right) &= \left(Fh, \int P_N F\, d\beta\right)\\ &= \left(Fh, \sum_{n=1}^{\infty} k_n \int ((P_N F)^* k_n, d\beta)\right)\\ &= \sum_{n=1}^{N} (k_n, Fh) \int (F^* k_n, d\beta) = \sum_{n=1}^{N} M_n (F^* k_n, h),\end{aligned}$$

and hence

$$\begin{aligned}\sum_{n=1}^{\infty} \int (M_n F^* k_n, d\beta) &= \lim_{N \to \infty} \sum_{n=1}^{N} \int (M_n F^* k_n, d\beta)\\ &= \lim_{N \to \infty} \int \left(F^* \int P_N F\, d\beta, d\beta\right). \qquad (8.6)\end{aligned}$$

So to finish the proof it suffices to consider

$$\begin{aligned}&\prec \int \left(F^* \int F\, d\beta, d\beta\right) - \int \left(F^* \int P_N F\, d\beta, d\beta\right) \succ_t\\ &= \int_0^t \left\| F_\tau^* \int_0^\tau (I - P_N)\, F\, d\beta \right\|^2 d\tau\\ &\leqslant \int_0^t \|F_\tau\|_{op}^2 \left\| \int_0^\tau (I - P_N)\, F\, d\beta \right\|^2 d\tau. \qquad (8.7)\end{aligned}$$

Since, $\|(I - P_N)\, Fh_n\|^2 \leqslant \|Fh_n\|^2$ and $\sum_{n=1}^{\infty} \|Fh_n\|^2 = \|F\|_{HS}^2 < \infty$, it follows by the dominated convergence theorem that

$$\prec \int (I - P_N)\, F\, d\beta \succ = \int \|(I - P_N)\, F\|_{HS}^2\, d\tau \to 0 \qquad \text{as} \quad N \to \infty, \quad (8.8)$$

and hence $\int (I - P_N)\, F\, d\beta$ converges to zero uniformly on compacts in probability. Therefore, the right side of Eq. (8.7) tends to zero as $N \to \infty$. This implies that $\int (F^* \int P_N F\, d\beta, d\beta) \to \int (F^* \int F\, d\beta, d\beta)$ in probability as $N \to \infty$. Equation (8.4) now follows from this limit and equations (8.5) and (8.6). Q.E.D.

8.3. *Backwards Itô Integrals*

Let $T > 0$ be fixed. For the moment suppose that V is a finite dimensional vector space, $\{X(t)\}_{t \geqslant 0}$ is a continuous V-valued process and

$\{A(t)\}_{t\geqslant 0}$ is continuous $End(V)$-valued process. Let $\pi = \{0 = t_0 < t_1 < t_1 < t_2 < \cdots < t_n = T\}$ denote a partition of $[0, T]$, $|\pi| \equiv \max_i |t_{i+1} - t_i|$. For $\tau = t_i \in \pi$, let $\tau+ \equiv t_{(i+1)}$ be the successor to τ in π. (By convention $t_{n+1} \equiv T$.) Then the *forward stochastic* and respectively *backwards stochastic integral* of A relative to X is

$$\int_0^t A\, dX \equiv \lim_{|\pi| \to 0} \sum_{\tau \in \pi} A(\tau)(X(t \wedge (\tau+)) - X(\tau \wedge t)) \tag{8.9}$$

and

$$\int_t^T A\, \overleftarrow{dX} \equiv \lim_{|\pi| \to 0} \sum_{\tau \in \pi} A(\tau+)(X(t \vee (\tau+)) - X(\tau \vee t)), \tag{8.10}$$

provided that limits exists in probability uniformly for t in $[0, T]$. For example, if A and X are semi-martingales then the above limit exists and

$$\int_t^T A\, \overleftarrow{dX} = \int_t^T A\, dX + \int_t^T dA\, dX,$$

where $\int_t^T A\, \overleftarrow{dX} = \int_0^T A\, \overleftarrow{dX} - \int_0^t A\, \overleftarrow{dX}$ and

$$\int_0^t dA\, dX \equiv \lim_{|\pi| \to 0} \sum_{\tau \in \pi} (A(t \wedge \tau+) - A(\tau))(X(t \wedge (\tau+)) - X(\tau \wedge t)) \tag{8.11}$$

is the joint quadratic variation between A and X. Set $A^T(t) \equiv A(T-t)$, $X^T(t) \equiv X(T-t) - X(T)$ and for each partition π of $[0, T]$ as above let π^T denote the partition

$$\pi^T = \{0 = T - t_n < T - t_{n-1} < \cdots < T - t_1 < T - t_0 = T\}.$$

Noting that $|\pi^T| = |\pi|$ and

$$\begin{aligned} &X(T - (T-t) \wedge (\tau+)) - X(T - (T-t) \wedge \tau) \\ &\quad = X(t \vee (T - \tau+)) - X(t \vee (T - \tau)), \end{aligned}$$

we have

$$\begin{aligned} \int_0^{T-t} A^T\, dX^T &\equiv \lim_{|\pi| \to 0} \sum_{\tau \in \pi} \sum_{\tau \in \pi} A(T-\tau)(X(T - (T-t) \wedge (\tau+)) \\ &\qquad - X(T - (T-t) \wedge \tau)) \\ &= \lim_{|\pi| \to 0} \sum_{\tau \in \pi^T} A(\tau+)[X(t \vee \tau) - X(t \vee \tau+)] \\ &= -\int_t^T A\, \overleftarrow{dX}. \end{aligned} \tag{8.12}$$

We will now use this last relationship as a definition for our infinite dimensional backwards Itô integrals. We now formulate the precise definition that is used in the body of this paper. As in the last subsection let $(\mathscr{W}, \{\mathscr{F}_t\}_{t\geqslant 0}, \mathscr{F}, P)$ be the filtered probability space and $\{\beta(t)\}_{t\geqslant 0}$ be the $\mathscr{L}(\mathfrak{g})$-valued Brownian motion as in the body of the text. Fix $T>0$ and set $\beta^T(t)\equiv\beta(T-t)-\beta(T)$ for $t\in[0,T]$, $\mathscr{F}^T$ denote the completion of the σ-algebra generated by $\{\beta^T(t): 0\leqslant t\leqslant T\}$ and $\mathscr{F}^T_t$ be the σ-algebra generated by $\{\beta^T(\tau): 0\leqslant\tau\leqslant t\}=\{\beta(\tau)-\beta(T): T-t\leqslant\tau\leqslant T\}$ augmented by the null sets of $\mathscr{F}^T$.

Definition 8.6. Suppose that $H(t)$ is a continuous (for simplicity) $H_0(\mathfrak{g})$-valued process such that $H(t)$ is $\mathscr{F}^T_{T-t}$-measurable. (Notice that $\mathscr{F}^T_{T-t}$ is the σ-algebra generated by $\{\beta(\tau)-\beta(T): t\leqslant\tau\leqslant T\}$ augmented by the null sets of $\mathscr{F}^T$.) Then $H^T(t)\equiv H(T-t)$ is a continuous process adapted to the filtration $\{\mathscr{F}^T_t\}_{0\leqslant t\leqslant T}$ and we define, for $0\leqslant t\leqslant T$, the backwards stochastic integral of H as

$$\int_t^T (H(\tau), \overleftarrow{d\beta}(\tau))\equiv-\int_0^{T-t}(H(T-\tau), d\beta^T(\tau)). \tag{8.13}$$

Remark 8.7. Notice that the backward Itô-integral defined in Eq. (8.13) inherits the basis the L^2-isometry property from the forward Itô integral, namely

$$E\left[\int_t^T (H(t), \overleftarrow{d\beta}(t))\right]^2=E\int_t^T \|H(t)\|^2\,dt \tag{8.14}$$

provided the right side of Eq. (8.14) is finite.

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