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HEAT KERNEL MEASURE ON LOOP AND PATH GROUPS

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Diffusions on the pinned loop and path group of a general (non-compact) Lie group are constructed. The laws of the endpoint processes satisfy heat equations related to second order differential operators associated to subspaces of the Cameron–Martin space.

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1. Introduction

1.1. *Background*

The existence and properties of Brownian motion on $\mathcal{L}(K)$, the pinned loop group of a compact Lie group K, have been studied in a number of papers starting with Refs. 20, 21 and then followed by Refs. 2, 3, 6, 8, 9, 12, 14, 16, 24. (This is only a partial list.) Similar results have been obtained on $\mathcal{W}(K)$, the pinned path group, in Ref. 11. Of fundamental importance in these constructions is the compactness of K; in particular, the fact that a compact-type Lie group admits an Ad_K-invariant inner product on its Lie algebra, $\mathfrak{k}$. In addition, several recent works (Refs. 1, 13 and 22) have constructed Brownian motion on the diffeomorphism group of the circle.

This paper extends results that can be found in Refs. 6, 8 and 9 to a more general type of diffusion. In particular, we eliminate the compactness assumption on K and any need of an Ad_K-invariant inner product on $\mathfrak{k}$. The existence of the heat kernel measures and the properties of finite-dimensional approximations summarized herein are important to the first author's analysis of the Taylor map on complex path groups in Ref. 4.

1.2. *Statement of results*

Let G be a connected real Lie group, $\mathfrak{g} = T_eG$ be its Lie algebra, Γ be a linearly independent[a] subset of $\mathfrak{g}$, and $V := \operatorname{span}(\Gamma) \subset \mathfrak{g}$. Further let

$$\beta(t,s) := \sum_{A\in\Gamma} \beta^A(t,s)A\,, \tag{1.1}$$

where $\{\beta^A\}_{A\in\Gamma}$ is a collection of independent $\mathbb{R}$-valued Brownian sheets or Brownian bridge sheets. To be more precise, if

$$k(\sigma,s) = s\wedge\sigma \text{ and } k_0(\sigma,s) = s\wedge\sigma - \sigma s \qquad \forall\, \sigma,s\in[0,1]\,, \tag{1.2}$$

then for each $A\in\Gamma$, $\{\beta^A(t,s) : s\in[0,1], t\geq 0\}$ is a mean zero continuous Gaussian process such that

$$\mathbb{E}[\beta^A(t,s)\beta^A(\tau,\sigma)] = \bar{k}(\sigma,s)(t\wedge\tau)\,, \tag{1.3}$$

where $\bar{k}(\sigma,s) = k(\sigma,s)$ in the Brownian case and $\bar{k}(\sigma,s) = k_0(\sigma,s)$ in the Brownian bridge case.

Suppose that $(\Omega,\mathcal{F},P)$ is a complete probability space on which the processes $\{\beta^A\}_{A\in\Gamma}$ are defined. For each $t\geq 0$, let $\mathcal{F}_t^0$ be the smallest sub-sigma-algebra of $\mathcal{F}$ such that $\beta^A(\tau,s)$ is measurable for all $s\in[0,1]$, $\tau\in[0,t]$ and $A\in\Gamma$. Let $\{\mathcal{F}_t\}_{t\geq 0}$ be the filtration which is the right continuous extension of the filtration $\{\mathcal{F}_t^0\}_{t\geq 0}$, augmented by all the P-null subsets of $\mathcal{F}$. This filtration then satisfies the "usual hypothesis," i.e. $\{\mathcal{F}_t\}_{t\geq 0}$ is right continuous and each $\mathcal{F}_t$ contains all of the P — null sets.

For g, $x\in G$, let $L_gx = gx$ and $R_gx = xg$. In addition, for $A\in\mathfrak{g}$ let $\tilde{A}$ be the unique left invariant vector field satisfying $\tilde{A}(e) = A\in\mathfrak{g}$. In general we will use "δ" for the Stratonovich differential and "d" for the Itô differential of a semimartingale. In particular, $\beta(\delta t,s)$ and $\beta(dt,s)$ denote the Stratonovich and Itô differentials, respectively, of the process $t\to\beta(t,s)$.

Define $\mathcal{W}(G)$ and $\mathcal{L}(G)\subset\mathcal{W}(G)$ to be the based path group and loop group respectively on G. Specifically,

$$\mathcal{W}(G) = \{\sigma : [0,1]\to G | \sigma \text{ is continuous and } \sigma(0) = e\}$$

and

$$\mathcal{L}(G) = \{\sigma : [0,1]\to G | \sigma \text{ is continuous and } \sigma(0) = \sigma(1) = e\}\,.$$

[a]See the Appendix to see why it is sufficient to consider only the case where $\Gamma\subset\mathfrak{g}$ is linearly independent.

To each partition,

$$\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_n < 1\}\,, \tag{1.4}$$

of [0, 1], we associate a projection map, $\pi_{\mathcal{P}} : \mathcal{W}(G) \to G^{\#(\mathcal{P})}$, defined by

$$\pi_{\mathcal{P}}(\sigma) = (\sigma(s_1), \sigma(s_2), \ldots, \sigma(s_n))\,, \tag{1.5}$$

where $\#(\mathcal{P}) := n$. We make $\mathcal{W}(G)$ and $\mathcal{L}(G)$ into measurable spaces by endowing each with the smallest σ-algebra for which all of the projection maps, $\{\pi_{\mathcal{P}} : \mathcal{P}$ a partition of [0, 1]$\}$, are measurable. The existence of $\mathcal{W}(G)$ and $\mathcal{L}(G)$ valued diffusions is given in the following theorem.

Theorem 1.1. *Suppose G is a Lie group and $\sigma_0 \in \mathcal{W}(G)$. Then there exists a continuous $\mathcal{F}_t$-adapted $\mathcal{W}(G)$-valued process, $\{\Sigma(t)\}_{t\geq 0}$, such that for each $s \in [0,1]$, $\Sigma(\cdot, s)$ solves the stochastic differential equation*:

$$\Sigma(\delta t, s) = L_{\Sigma(t,s)*}\beta(\delta t, s) \quad \textit{with } \Sigma(0,s) = \sigma_0(s)\,. \tag{1.6}$$

More precisely,

$$\Sigma(\delta t, s) = \sum_{A\in\Gamma} \tilde{A}(\Sigma(t,s))\beta^A(\delta t, s) \quad \textit{with } \Sigma(0,s) = \sigma_0(s)\,. \tag{1.7}$$

We will prove Theorem 1.1 by first proving it in the case where G is the general linear group, $\mathrm{GL}(n,\mathbb{R})$, (see Sec. 2), then in the case that G is a (not necessarily closed) Lie sub-group of $\mathrm{GL}(n,\mathbb{R})$ (see Sec. 3), and then finally for general G (see Sec. 4).

Remark 1.1. When $\beta(t,s)$ is a $\mathfrak{g}$-valued Brownian bridge sheet, then the process given by Theorem 1.1 is $\mathcal{L}(G)$-valued. In this case we will denote Σ by Σ^0.

Definition 1.1. When Σ and Σ^0 are as in Theorem 1.1 and Remark 1.1 with $\sigma_0(s) \equiv e \in G$, let ν_t and ν_t^0 be the measures on $\mathcal{W}(G)$ and $\mathcal{L}(G)$ which are the laws of $\Sigma(t,\cdot)$ and $\Sigma^0(t,\cdot)$ respectively. Given a bounded or non-negative measurable map, $f : \mathcal{W}(G) \to \mathbb{R}$, we will use the following definition;

$$\nu_t(f) := \int_{\mathcal{W}(G)} f(g)d\nu_t(g) = \mathbb{E}f(\Sigma(t,\cdot)) \tag{1.8}$$

and

$$\nu_t^0(f) := \int_{\mathcal{W}(G)} f(g)d\nu_t^0(g) = \mathbb{E}f(\Sigma^0(t,\cdot))\,. \tag{1.9}$$

Because of Proposition 1.1 below, we will refer to ν_t and ν_t^0 as *heat kernel measures.*

Let $(\cdot,\cdot)_V$ denote the unique inner product on V for which Γ is an orthonormal basis and for a path $h : [0,1] \to V$, let

$$\langle h, h\rangle_{H(V)} := \begin{cases} \displaystyle\int_0^1 |h'(s)|_V^2 ds\,, & \text{if } h \text{ is absolutely continuous} \\ \infty\,, & \text{otherwise}\,. \end{cases}$$

We define the "horizontal" *Cameron–Martin spaces* as

$$H(V) = \{h : [0,1] \to V | h(0) = 0 \text{ and } (h,h)_{H(V)} < \infty\}$$

and

$$H_0(V) = \{h \in H(V) | h(1) = 0\}\,.$$

Given h, $l \in H(V)$, we can define a real inner product on $H(V)$ by

$$\langle h, l\rangle_{H(V)} = \int_0^1 (h'(s), l'(s))_V ds\,.$$

With this inner product, $H_0(V) \subset H(V)$ are Hilbert spaces. We will use S_0 and S to denote orthonormal bases of the Hilbert spaces $(H_0(V), \langle\cdot,\cdot\rangle_{H(V)})$ and $(H(V), \langle\cdot,\cdot\rangle_{H(V)})$ respectively.

Definition 1.2. A function $f : \mathcal{W}(G) \to \mathbb{C}$ is called a *smooth cylinder function* if there exist a partition $\mathcal{P}$ of $[0,1]$ and a function $F \in C^\infty(G^{\#(\mathcal{P})})$ such that $f = F \circ \pi_{\mathcal{P}}$.

For $h \in H(V)$, $g \in \mathcal{W}(G)$, and $f : \mathcal{W}(G) \to \mathbb{C}$ a smooth cylinder function, let

$$\tilde{h} f(g) := \frac{d}{dt}|_0 f(g \cdot e^{th})\,, \tag{1.10}$$

where $(g \cdot e^{th})(s) := g(s)e^{th(s)}$ for all $s \in [0,1]$. In Sec. 5, we will show that the linear operators

$$L_{H(V)} f := \sum_{h\in S} \tilde{h}^2 f \text{ and } L_{H_0(V)} f := \sum_{h\in S_0} \tilde{h}^2 f \tag{1.11}$$

on smooth cylinder functions on $\mathcal{W}(G)$ and $\mathcal{L}(G)$ respectively are the generators of our diffusions given by Theorem 1.1. As will be shown in Sec. 5.1, the heat kernel measures, ν_t and ν_t^0, satisfy (in the distributional sense) the following "heat" equations.

Proposition 1.1. *If $f = F \circ \pi_{\mathcal{P}}$ is a smooth cylinder function, then*

$$\frac{\partial}{\partial t}\nu_t(f) = \frac{1}{2}\nu_t(L_{H(V)} f) \text{ with } \lim_{t\downarrow 0} \nu_t(f) = f(\mathbf{e}) \tag{1.12}$$

and

$$\frac{\partial}{\partial t}\nu_t^0(f) = \frac{1}{2}\nu_t^0(L_{H_0(V)} f) \text{ with } \lim_{t\downarrow 0} \nu_t^0(f) = f(\mathbf{e})\,, \tag{1.13}$$

where $\mathbf{e}$ is the identity path in $\mathcal{W}(G)$.

2. $\mathcal{W}(\mathrm{GL}(n,\mathbb{R}))$-valued Diffusions

We will first prove Theorem 1.1 in the special case where $G := \mathrm{GL}(n,\mathbb{R})$ is the general linear group of $n \times n$-real invertible matrices. Let $\mathrm{gl}(n,\mathbb{R}) = \mathrm{Lie}(\mathrm{GL}(n,\mathbb{R}))$ be the Lie algebra of $\mathrm{GL}(n,\mathbb{R})$ consisting of all real $n \times n$ matrices. We will make $\mathrm{gl}(n,\mathbb{R})$ into a Hilbert space with the aid of the Hilbert–Schmidt inner product and norm given by $(A,B) := \mathrm{tr}(A^{\mathrm{tr}}B)$ and $|A| := \sqrt{\mathrm{tr}(A^{\mathrm{tr}}A)}$ respectively. It is easily verified that $|I| = \sqrt{n}$ where I is the identity matrix in $\mathrm{GL}(n,\mathbb{R})$ and $|AB| \le |A||B|$ for all A, $B \in \mathrm{gl}(n,\mathbb{R})$.

Remark 2.1. Throughout the paper, $C_p(\Gamma',T)$ will be used to denote a generic finite constant which may vary from line to line. However, $C_p(\Gamma',T)$ will always only depend on $p \in [2,\infty)$, $T \in [0,\infty)$, and Γ' where Γ' is a finite subset of $\mathfrak{g}$.

Theorem 2.1. *For each $s \in [0,1]$, let $g(t,s)$ be the* $\mathrm{GL}(n,\mathbb{R})$*-valued solution to the linear stochastic differential equation,*

$$g(dt,s) = g(t,s)\beta(\delta t,s) \text{ with } g(0,s) = I\,. \tag{2.1}$$

Then for any $T < \infty$ and $p \in [2,\infty)$,

$$\mathbb{E}[\max_{0\le t\le T} |g(t,s) - g(t,\sigma)|^p] \le C_p(\Gamma,T)|s-\sigma|^{p/2} \ \forall\ \sigma, s \in [0,1]\,, \tag{2.2}$$

and

$$\mathbb{E}|g(t,s) - g(\tau,\sigma)|^p \le C_p(\Gamma,T)[|t-\tau|^{p/2} + |s-\sigma|^{p/2}] \tag{2.3}$$

for all σ, $s \in [0,1]$ and t, $\tau \in [0,T]$. Both of these estimates hold, with the same constants, when g is replaced by the inverse process, g^{-1}.

The proof of Theorem 2.1 will be completed in Sec. 2.2 below after first proving some preliminary results. The following result is a simple corollary of Theorem 2.1 along with Kolmogorov's continuity criteria (see for example Theorem 1.4.1 of Ref. 19 and Corollary 1.2 of Ref. 25).

Corollary 2.1. *Keeping the same definition as in Theorem* 2.1, *there is a jointly continuous version of g solving Eq.* (2.1). *Moreover, for any $\alpha \in (0,1/2)$ there exists a random variable, $C_\alpha < \infty$ a.s., such that*

$$\sup_{0\le t\le T} |g(t,s) - g(t,\sigma)| \le C_\alpha |s-\sigma|^\alpha \tag{2.4}$$

and

$$|g(t,s) - g(\tau,\sigma)| \le C_\alpha[|t-\tau|^\alpha + |s-\sigma|^\alpha]\,. \tag{2.5}$$

2.1. *Preliminary results*

Let us begin by recalling a standard Brownian semi-martingale estimate, see Eq. (2.7) below. First we need some definition. Let Γ be an arbitrary linearly independent subset of $\mathrm{gl}(n,\mathbb{R})$, $\{B^A\}_{A\in\Gamma}$ be independent Brownian motions, and B_t be the $\mathrm{gl}(n,\mathbb{R})$-valued Brownian motion defined by,

$$B_t := \sum_{A\in\Gamma} B^A(t)A\,. \tag{2.6}$$

For any $\mathrm{gl}(n,\mathbb{R})$-valued process Y_t, let $Y_t^* := \sup_{s\le t}|Y_t|$. Suppose $\tau \to Q_\tau \in \mathrm{End}(V,\mathrm{gl}(n,\mathbb{R}))$ and $\tau \to \alpha_\tau \in \mathrm{gl}(n,\mathbb{R})$ are continuous adapted processes and

$$Y_t := \int_0^t Q_\tau dB_\tau + \int_0^t \alpha_\tau d\tau \in \mathrm{gl}(n,\mathbb{R})\,.$$

Then for each $p\in[2,\infty)$ there exists $c_p<\infty$ such that

$$\mathbb{E}(Y_t^*)^p \le c_p\left\{\mathbb{E}\left(\int_0^t \|Q_\tau\|_\Gamma^2 d\tau\right)^{p/2} + \mathbb{E}\left(\int_0^t |\alpha_\tau| d\tau\right)^p\right\}, \tag{2.7}$$

where

$$\|Q_\tau\|_\Gamma^2 = \sum_{A\in\Gamma}|Q_\tau(A)|^2\,, \tag{2.8}$$

see for example Proposition 9.2 of Ref. 7. For $\alpha\ge 1$ and $u_\tau\ge 0$, Jensen's inequality implies,

$$\left(\int_0^t u_\tau d\tau\right)^\alpha = t^\alpha\left(\int_0^t u_\tau\frac{d\tau}{t}\right)^\alpha \le t^\alpha\int_0^t u_\tau^\alpha\frac{d\tau}{t} = t^{\alpha-1}\int_0^t u_\tau^\alpha d\tau\,.$$

Combining this estimate with Eq. (2.7) gives,

$$\mathbb{E}(Y_t^*)^p \le c_p\left\{t^{p/2-1}\mathbb{E}\int_0^t \|Q_\tau\|_\Gamma^p d\tau + t^{p-1}\mathbb{E}\int_0^t |\alpha_\tau|^p d\tau\right\}. \tag{2.9}$$

In applying Eq. (2.9), Q_τ will be of the form, $Q_\tau := L_{x_\tau}R_{y_\tau}$ with x_τ and y_τ in $\mathrm{gl}(n,\mathbb{R})$. The $\|\cdot\|_\Gamma$-norm of such a Q_τ may easily be estimated as:

$$\|Q_\tau\|_\Gamma^2 = \sum_{A\in\Gamma}|x_\tau A y_\tau|^2 \le C(\Gamma)|x_\tau|^2|y_\tau|^2\,, \tag{2.10}$$

where

$$C(\Gamma) := \sum_{A\in\Gamma}|A|^2\,. \tag{2.11}$$

Now let $\{g_t\}_{t\ge0}$ be the $\mathrm{GL}(n,\mathbb{R})$-valued solution to the stochastic differential equation;

$$dg_t = g_t\delta B_t \text{ with } g_0 = I\,. \tag{2.12}$$

The integrated Itô form of Eq. (2.12) is

$$g_t - I = \int_0^t g_\tau dB_\tau + \frac{1}{2}\int_0^t g_\tau [dB_\tau]^2\,, \tag{2.13}$$

where

$$[dB_t]^2 = \left(\sum_{A\in\Gamma} dB_t^A A\right)^2 = \sum_{A\in\Gamma} A^2 dt\,.$$

It is standard, see Theorem 5.2.9 of Ref. 18 for example, that this linear stochastic differential equation has a unique global strong solution.

The next Proposition 2.1 summarizes some well-known estimates on the process $\{g_t\}_{t\geq 0}$ solving Eq. (2.12). Since we will need to keep fairly careful track of the constants appearing in these estimates, we will sketch the proof of Proposition 2.1. It will be convenient in what follows to let

$$K(p,T,\Gamma) := 2^{p-1}c_p[C(\Gamma)^{p/2}T^{\frac{p}{2}} + 2^{-p}T^pC(\Gamma)^p]\,, \tag{2.14}$$

where c_p is as in Eq. (2.7) and $C(\Gamma)$ is defined in Eq. (2.11).

Proposition 2.1. *Let g be the process given by Eq.* (2.12). *Then for each $p \in [2,\infty)$,*

$$\sup_{t\leq T}\mathbb{E}[|g_t|^p] \leq 2^{p-1}n^{p/2}e^{K(p,T,\Gamma)} \tag{2.15}$$

and, for all s, $t \geq 0$,

$$\mathbb{E}[|g_t - g_s|^p] \leq 2^{p-1}n^pK(p,|t-s|,\Gamma)e^{K(p,s\wedge t,\Gamma)+K(p,|t-s|,\Gamma)}\,. \tag{2.16}$$

Similarly, the inverse process, g_t^{-1}, satisfies the exact same estimates in Eqs. (2.15) *and* (2.16) *with g replaced by g^{-1}.*

Proof. Let $T < \infty$ be fixed and throughout suppose that $t, \tau \in [0,T]$. Applying the estimate in Eq. (2.9) with $Q_\tau := L_{g_\tau}$ and $\alpha_\tau := \frac{1}{2}g_\tau\sum_{A\in\Gamma}A^2$, using the estimates

$$|\alpha_\tau| \leq \frac{1}{2}C(\Gamma)|g_\tau| \text{ and } \|Q_\tau\|_\Gamma^2 \leq C(\Gamma)|g_\tau|^2\,,$$

implies

$$\mathbb{E}(g_\cdot - I)_t^{*p} \leq c_p[C(\Gamma)^{p/2}t^{(\frac{p}{2}-1)} + t^{p-1}2^{-p}C(\Gamma)^p]\left(\int_0^t \mathbb{E}|g_\tau|^p d\tau\right)\,. \tag{2.17}$$

Since

$$|g_t|^p \leq [|I| + (g_\cdot - I)_t^*]^p \leq 2^{p-1}[n^{p/2} + (g_\cdot - I)_t^{*p}]$$

it follows from Eq. (2.17) that

$$\mathbb{E}|g_t|^p \leq 2^{p-1}n^{p/2} + 2^{p-1}c_p[C(\Gamma)^{p/2}T^{(\frac{p}{2}-1)} + T^{p-1}2^{-p}C(\Gamma)^p]\left(\int_0^t \mathbb{E}|g_\tau|^p d\tau\right)$$

and, assuming $\sup_{t\le T}\mathbb{E}|g_t|^p<\infty$, it follows by Gronwall's inequality that

$$\begin{aligned}\mathbb{E}|g_t|^p &\le 2^{p-1}n^{p/2}\cdot\exp(2^{p-1}c_p[C(\Gamma)^{p/2}T^{(\frac{p}{2}-1)}+T^{p-1}2^{-p}C(\Gamma)^p]t)\\ &\le 2^{p-1}n^{p/2}\cdot e^{K(p,T,\Gamma)}\,. \end{aligned}\tag{2.18}$$

By stopping g and B at first exit time of g_t from an increasing sequence of compact subsets which exhaust $\mathrm{GL}(n,\mathbb{R})$, we can easily remove the assumption that $\sup_{t\le T}\mathbb{E}|g_t|^p<\infty$ used to derive Eq. (2.18). Feeding Eq. (2.18) back into Eq. (2.17) at $t=T$ then implies

$$\mathbb{E}(g_\cdot-I)^{*p}_T\le n^{p/2}K(p,T,\Gamma)e^{K(p,T,\Gamma)}\,.\tag{2.19}$$

Now suppose that $s\in(0,\infty)$, $t=s+T$ for some $T>0$, and let $u_\tau:=g_s^{-1}g_{\tau+s}$ for $\tau\ge 0$. Since

$$du_\tau=u_\tau\delta_\tau[B_{(\tau+s)}-B_s]\text{ with }u_0=I\,,$$

and $\{B_{\tau+s}-B_s\}_{\tau\ge0}$ has the same law as $\{B_\tau\}_{\tau\ge0}$, it follows from Eq. (2.19) that

$$\mathbb{E}|g_s^{-1}g_{s+T}-I|^p=\mathbb{E}|u_T-I|^p\le n^{p/2}K(p,T,\Gamma)e^{K(p,T,\Gamma)}\,.$$

Since

$$|g_t-g_s|=|g_{s+T}-g_s|=|g_s(g_s^{-1}g_{s+T}-I)|\le|g_s|\cdot|(g_s^{-1}g_{s+T}-I)|$$

and g_s is independent of $g_s^{-1}g_{s+T}-I$, we find

$$\begin{aligned}\mathbb{E}|g_t-g_s|^p&\le\mathbb{E}|g_s|^p\cdot\mathbb{E}|(g_s^{-1}g_{s+T}-I)|^p\\ &\le 2^{p-1}n^{p/2}\cdot e^{K(p,s,\Gamma)}\cdot n^{p/2}K(p,T,\Gamma)e^{K(p,T,\Gamma)}\\ &=2^{p-1}n^pK(p,T,\Gamma)e^{K(p,s,\Gamma)+K(p,T,\Gamma)}\end{aligned}$$

from which Eq. (2.16) easily follows.

Since

$$\delta g_t^{-1}=-g_t^{-1}(\delta g_t)g_t^{-1}=-g_t^{-1}(g_t\delta B_t)g_t^{-1}=-\delta B_tg_t^{-1}\,,$$

it follows that

$$g_t^{-1}-I=-\int_0^t dB_\tau g_\tau+\frac{1}{2}\int_0^t[dB_\tau]^2g_\tau\,.$$

The process, $Q_\tau:=-R_{g_\tau}$ satisfies (see Eq. (2.10)) the same estimates as L_{g_τ}. Therefore, by the same methods as above, $g_\cdot^{-1}$ also satisfies the estimates in Eqs. (2.15) and (2.16). $\square$

2.2. *Proof of Theorem* 2.1

For each $s \in [0,1]$, let $g(t,s) \in \mathrm{GL}(n,\mathbb{R})$ solve the stochastic differential equation

$$g(dt,s) = g(t,s)\beta(\delta t,s) \text{ with } g(0,s) = I\,. \tag{2.20}$$

For an $s \in (0,1)$ (fixed), let

$$\Gamma_s := \left\{\tilde{A} := \sqrt{\bar{k}(s,s)}A\right\}_{A\in\Gamma}\,, \quad \text{and } B_t^{\tilde{A}} = \frac{\beta^A(t,s)}{\sqrt{\bar{k}(s,s)}}\,. \tag{2.21}$$

With this definition, $\{B_t^{\tilde{A}}\}_{\tilde{A}\in\Gamma_s}$ are independent Brownian motions such that

$$B_t := \beta(t,s) = \sum_{\tilde{A}\in\Gamma_s} B_t^{\tilde{A}}\tilde{A}\,. \tag{2.22}$$

Thus we may apply Eq. (2.16) with Γ replaced by Γ_s to estimate $g(t,s) - g(\tau,s)$ as,

$$\mathbb{E}[|g(t,s) - g(\tau,s)|^p] \le 2^{p-1}n^pK(p,|t-\tau|,\Gamma_s)e^{K(p,\tau\wedge t,\Gamma_s)+K(p,|t-\tau|,\Gamma_s)}\,. \tag{2.23}$$

Using $C(\Gamma_s) = \bar{k}(s,s)C(\Gamma) \le C(\Gamma)$ ($C(\Gamma)$ was defined in Eq. (2.11)) and the definition of K in Eq. (2.14), we find

$$\mathbb{E}[|g(t,s) - g(\tau,s)|^p] \le C_p(\Gamma,T)|t-\tau|^{p/2}\ \forall\ s \in [0,1] \text{ and } t,\tau \in [0,T]\,. \tag{2.24}$$

Now suppose $0 < \sigma < s < 1$ and let

$$u_t := g(t,s)g(t,\sigma)^{-1} \text{ and} \tag{2.25}$$

$$B_t := \beta(t,s) - \beta(t,\sigma) \tag{2.26}$$

in which case u_t solves

$$du_t = g(t,s)[\beta(\delta t,s) - \beta(\delta t,\sigma)]g(t,\sigma)^{-1} = u_t Ad_{g(t,\sigma)}\delta B_t\,. \tag{2.27}$$

Since

$$\begin{aligned} d_t[Ad_{g(t,\sigma)}]dB_t &= Ad_{g(t,\sigma)}ad_{d_t\beta(t,\sigma)}dB_t \\ &= \sum_{A\in\Gamma} Ad_{g(t,\sigma)}ad_A Ad_t\beta^A(t,\sigma)d_tB^A = 0\,, \end{aligned}$$

we find

$$\begin{aligned} d_t[u_tAd_{g(t,\sigma)}]dB_t &= \frac{1}{2}(d_tu_t)Ad_{g(t,\sigma)}dB_t \\ &= \frac{1}{2}[u_tAd_{g(t,\sigma)}dB_t][Ad_{g(t,\sigma)}dB_t] \\ &= \frac{1}{2}\sum_{A\in\Gamma} u_tAd_{g(t,\sigma)}A^2(dB_t^A)^2 \\ &= \frac{1}{2}\sum_{A\in\Gamma} g(t,s)A^2g(t,\sigma)^{-1}(dB_t^A)^2\,, \end{aligned} \tag{2.28}$$

where $B_t^A := \beta^A(t,s) - \beta^A(t,\sigma)$. If we let

$$F(\sigma,s) = [\bar{k}(s,s) + \bar{k}(\sigma,\sigma) - 2\bar{k}(\sigma,s)], \tag{2.29}$$

$$\Gamma_{\sigma,s} := \{\hat{A} := \sqrt{F(\sigma,s)}A : A \in \Gamma\}, \quad \text{and} \tag{2.30}$$

$$B_t^{\hat{A}} = \frac{\beta^A(t,s) - \beta^A(t,\sigma)}{\sqrt{F(\sigma,s)}} = \frac{B_t^A}{\sqrt{F(\sigma,s)}}, \tag{2.31}$$

then, by checking covariances, $\{B_t^{\hat{A}}\}_{\hat{A}\in\Gamma_{\sigma,s}}$ are independent Brownian motions such that

$$B_t = \beta(t,s) - \beta(t,\sigma) = \sum_{\hat{A}\in\Gamma_{\sigma,s}} B_t^{\hat{A}}\hat{A}. \tag{2.32}$$

(A simple direct computation shows that $F(\sigma,s) > 0$ if $\sigma \neq s$.) Furthermore, since $B_t^A = \sqrt{F(\sigma,s)}B_t^{\hat{A}}$, it follows that $(dB_t^A)^2 = F(\sigma,s)dt$. From this observation and Eq. (2.28), the integrated Itô form of Eq. (2.27) is given by

$$u_t - I = \int_0^t L_{g(\tau,s)}R_{g(\tau,\sigma)^{-1}}dB_\tau + \frac{1}{2}F(\sigma,s)\int_0^t \sum_{A\in\Gamma} g(\tau,s)A^2 g(\tau,\sigma)^{-1}d\tau. \tag{2.33}$$

Using the decomposition of B_t in Eq. (2.32), we may apply the estimate in Eq. (2.9) to Eq. (2.33) to find

$$\begin{aligned}\mathbb{E}[(u-I)_t^{*p}] \le c_p\Bigg\{ & t^{p/2-1}\mathbb{E}\int_0^t \|L_{g(\tau,s)}R_{g(\tau,\sigma)^{-1}}\|_{\Gamma_{\sigma,s}}^p d\tau \\ & + t^{p-1}\mathbb{E}\int_0^t \left|\frac{1}{2}F(\sigma,s)\sum_{A\in\Gamma} g(\tau,s)A^2 g(\tau,\sigma)^{-1}\right|^p d\tau\Bigg\}.\end{aligned} \tag{2.34}$$

Since

$$\left|\sum_{A\in\Gamma} g(\tau,s)A^2 g(\tau,\sigma)^{-1}\right| \le C(\Gamma)|g(\tau,s)||g(\tau,\sigma)^{-1}|,$$

$C(\Gamma_{\sigma,s}) = F(\sigma,s)C(\Gamma)$ and by Eq. (2.10),

$$\|L_{g(\tau,s)}R_{g(\tau,\sigma)^{-1}}\|_{\Gamma_{\sigma,s}}^2 \le C(\Gamma)F(\sigma,s)|g(\tau,s)|^2|g(\tau,\sigma)^{-1}|^2,$$

it follows from Eq. (2.34) that

$$\mathbb{E}((u-I)_t^*)^p \le c_p(C(\Gamma)^{p/2}F^{p/2}(\sigma,s)t^{p/2-1} + 2^{-p}C(\Gamma)^pF^p(\sigma,s)t^{p-1})N \tag{2.35}$$

where

$$N := \mathbb{E}\int_0^t |g(\tau,s)|^p|g(\tau,\sigma)^{-1}|^p d\tau.$$

By Hölder's inequality and the estimate in Eq. (2.15),

$$N \le C(p,T) := \int_0^T (\mathbb{E}[|g(\tau,s)|^{2p}])^{1/2}(\mathbb{E}[|g(\tau,\sigma)^{-1}|^{2p}])^{1/2}d\tau < \infty$$

which combined with Eq. (2.35) implies

$$\mathbb{E}[(u-I)_t^{*p}] \le c_p C(p,T)(C(\Gamma)^{p/2}F^{p/2}(\sigma,s)t^{p/2-1} + 2^{-p}C(\Gamma)^p F^p(\sigma,s)t^{p-1})\,. \tag{2.36}$$

Since

$$\begin{aligned}|g(t,s)-g(t,\sigma)| &= |[g(t,s)g(t,\sigma)^{-1}-I]g(t,\sigma)| \\ &\le |g(t,s)g(t,\sigma)^{-1}-I||g(t,\sigma)| \\ &= |u_t - I||g(t,\sigma)|\,,\end{aligned}$$

it follows that

$$[g(\cdot,s)-g(\cdot,\sigma)]_T^{*p} \le [u_\cdot - I]_T^{*p} g_T^{*p}(\cdot,\sigma)\,. \tag{2.37}$$

By Hölder's inequality, the estimate in Eq. (2.36), and Proposition 2.1 we have

$$\begin{aligned}(\mathbb{E}[g(\cdot,s)-g(\cdot,\sigma)]_T^{*p})^2 &\le \mathbb{E}[u-I]_T^{*2p}\cdot \mathbb{E}g_T^{*2p}(\cdot,\sigma) \\ &\le C_p(\Gamma,T)[F(\sigma,s)^p + F(\sigma,s)^{2p}]\,. \end{aligned}\tag{2.38}$$

In each of the two cases considered (Brownian sheets and Brownian bridge sheets), we have $|\frac{\partial}{\partial s}F(\sigma,s)| \le 2$ and $F(\sigma,\sigma)=0$ and therefore,

$$F(\sigma,s) = \left|\int_\sigma^s \frac{\partial}{\partial r}F(\sigma,r)dr\right| \le 2|s-\sigma|\,. \tag{2.39}$$

Combining the estimates in Eqs. (2.38) and (2.39) then implies

$$\begin{aligned}\mathbb{E}[g(\cdot,s)-g(\cdot,\sigma)]_T^{*p} &\le C_p(\Gamma,T)[|s-\sigma|^p + |s-\sigma|^{2p}]^{1/2} \\ &\le C_p(\Gamma,T)|s-\sigma|^{p/2}\,. \end{aligned}\tag{2.40}$$

Finally, let $0 \le \tau \le T$ and $0 \le \sigma \le s \le 1$. From Eqs. (2.24), (2.40), and the triangle inequality we have,

$$\begin{aligned}\|g(t,s)-g(\tau,\sigma)\|_{L^p} &\le \|g(t,s)-g(\tau,s)\|_{L^p} + \|g(\tau,s)-g(\tau,\sigma)\|_{L^p} \\ &\le C_p(\Gamma,T)[|t-\tau|^{1/2} + |s-\sigma|^{1/2}]\end{aligned}$$

which implies the estimate in Eq. (2.3).

3. Matrix Groups

Though G may not have a finite dimensional real representation, Ado's Theorem (p. 199 of Ref. 17) states that $\mathfrak{g}$ has a faithful representation, $\varphi : \mathfrak{g} \to \mathfrak{g}_0$, where $\mathfrak{g}_0$ is a matrix Lie sub-algebra of $\mathrm{gl}(n,\mathbb{R})$ for some $n \in \mathbb{N}$. Let $G_0 \subset \mathrm{GL}(n,\mathbb{R})$ be the unique connected Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$ with $\mathrm{Lie}(G_0) = \mathfrak{g}_0$. If we let $\mathcal{D} := \{\tilde{A} : A \in \mathfrak{g}_0 \subset \mathrm{gl}(n,\mathbb{R})\}$, then $\mathcal{D}$ is an involutive distribution and G_0 may be described as the unique maximal $\mathcal{D}$-integral manifold in $\mathrm{GL}(n,\mathbb{R})$ which contains $I \in \mathrm{GL}(n,\mathbb{R})$, see Theorem 3.19 of Ref. 26.

Remark 3.1. While G_0 is a subgroup, it is not necessarily closed as a subset of $\mathrm{GL}(n,\mathbb{R})$. See Sec. 3.8 of Ref. 15 for an example of such a matrix group.

Definition 3.1. Let τ_0 be the manifold topology on G_0 and τ_i be the topology on G_0 which is inherited from $\mathrm{GL}(n,\mathbb{R})$. In general, τ_i may be a proper subset of τ_0.

We give $\mathrm{GL}(n,\mathbb{R})$ a Riemannian structure by extending the inner product, $(\cdot,\cdot)$, on $\mathrm{gl}(n,\mathbb{R})$ to a right invariant Riemannian metric on $\mathrm{GL}(n,\mathbb{R})$. This metric induces a right-invariant distance on $\mathrm{GL}(n,\mathbb{R})$ defined by,

$$d_{\mathrm{GL}(n,\mathbb{R})}(x,y) = \inf_{\sigma} \int_0^1 |\sigma'(s)\sigma^{-1}(s)|ds\,, \tag{3.1}$$

where the infimum is taken over piecewise C^1-paths, $\sigma : [0,1] \to \mathrm{GL}(n,\mathbb{R})$, such that $\sigma(0) = x$ and $\sigma(1) = y$.

Let $\exp : \mathrm{gl}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ be the matrix exponential map defined by,

$$\exp(A) := e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \text{ for all } A \in \mathrm{gl}(n,\mathbb{R}) \tag{3.2}$$

and

$$\log(x) = -\sum_{n=1}^{\infty} \frac{1}{n}(I-x)^n \text{ for } |I-x| < 1 \tag{3.3}$$

be its local inverse.

Lemma 3.1. *There exists connected open neighborhoods,* $N \subset \mathfrak{g}_0$ *and* $W \subset \mathrm{GL}(n,\mathbb{R})$, *of* $0 \in \mathfrak{g}_0$ *and* $I \in \mathrm{GL}(n,\mathbb{R})$ *respectively such that* $\exp(N)$ *is the* τ_0-*connected component of* $G_0 \cap W$ *containing* $I \in G_0$.

Proof. Let $\mathfrak{g}_0'$ be a complementary subspace to $\mathfrak{g}_0$ in $\mathrm{gl}(n,\mathbb{R})$ and define $\psi : \mathfrak{g}_0 \times \mathfrak{g}_0' \to \mathrm{GL}(n,\mathbb{R})$ by; $\psi(A,B) := e^B e^A$. Then it is well known (see for example Theorem 1.5.3 of Ref. 10) that

$$\frac{d}{dt}|_0 \psi(A+tA',B) = L_{\psi(A,B)*} \int_0^1 e^{-s\,ad_A} A' ds \in L_{\psi(A,B)*}\mathfrak{g}_0 = \mathcal{D}_{\psi(A,B)}\,.$$

Hence for each $B \in \mathfrak{g}_0'$ and a sufficiently small open neighborhood, N, of $0 \in \mathfrak{g}_0$, $N \ni A \to e^B e^A$ is a $\mathcal{D}$-integral sub-manifold in $\mathrm{GL}(n,\mathbb{R})$.

Let us choose connected open neighborhoods, N and N', of 0 in $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ respectively such that $\psi : N \times N' \to \mathrm{GL}(n,\mathbb{R})$ is a diffeomorphism onto $W := \psi(N \times N')$ — an open connected neighborhood of $I \in \mathrm{GL}(n,\mathbb{R})$. We have just shown that $z := (x,y) := \psi|_{N\times N'}^{-1}$ is a chart on $W \subset \mathrm{GL}(n,\mathbb{R})$ such that $\{y = B\}$ with $B \in N'$ are integral submanifolds of $\mathfrak{D}$.

Let $\mathcal{C}$ be the τ_0-connected component of $G_0 \cap W$ which contains $I \in G_0$. Since $G_0 \cap W$ is τ_0-locally path connected, $\mathcal{C}$ is τ_0-open (and closed in general) in $G_0 \cap W$ and therefore $\mathcal{C}$ is a connected integral submanifold of $\mathcal{D}$. So for $h \in \mathcal{C}$, $T_h\mathcal{C} = L_{h*}\mathfrak{g}_0 = \mathcal{D}_h$ and therefore, $dy(T_h\mathcal{C}) = 0$. Now $y|_{\mathcal{C}}$ is constant because $\mathcal{C}$ is also connected in the weaker topology, τ_i, on $G_0 \cap W$ inherited from $\mathrm{GL}(n,\mathbb{R})$. Since $I \in \mathcal{C}$ and $y(I) = 0$ it follows that $\mathcal{C} \subset \{y = 0\} = \exp(N)$. As $\exp : N \to (G_0 \cap W, \tau_0)$ continuous map, $\exp(N) \subset G_0 \cap W$ is a τ_0-connected set containing I. Therefore $\exp(N) \subset \mathcal{C}$ and thus $\mathcal{C} = \exp(N)$. $\square$

Proposition 3.1. *There exists a connected open neighborhood, θ, of $0 \in \mathrm{gl}(n,\mathbb{R})$ such that*:

(i) $\tilde{\theta} := \exp(\theta)$ *is a connected open neighborhood of $I \in \mathrm{GL}(n,\mathbb{R})$*,
(ii) $\exp : \theta \to \tilde{\theta}$ *is a diffeomorphism*,
(iii) *there are constants C_1, $C_2 > 0$ such that*

$$C_1|y - x| \leq d_{\mathrm{GL}(n,\mathbb{R})}(x, y) \leq C_2|y - x| \tag{3.4}$$

for all x, $y \in \tilde{\theta}$,
(iii) *there exists $C < \infty$ such that*

$$|\log(x)| \leq C|I - x| \text{ for all } x \in \tilde{\theta}\,, \tag{3.5}$$

(v) *and the τ_0-connected component, $\mathcal{C}$, of $G_0 \cap \tilde{\theta}$ containing I is* $\exp(\theta \cap \mathfrak{g}_0)$.

Proof. Items (i)–(iv) will hold for any sufficiently small open neighborhood, θ, of $0 \in \mathrm{GL}(n,\mathbb{R})$. This is true for items (i) and (ii) by the inverse function theorem. Item (iii) holds because $\mathrm{GL}(n,\mathbb{R})$ is an open subset of $\mathrm{gl}(n,\mathbb{R})$ and the fact that the metric space topology coincides with topology determined by the differentiable structure (see Corollary I.6.1 of Ref. 5). Item (iv) holds by a simple estimate of the power series expansion for $\log(x)$ in Eq. (3.3).

For item (v), we take $\tilde{\theta} = W = \psi(N \times N')$ as in the proof of Lemma 3.1 with N and N' sufficiently small so that $\theta := \log(\tilde{\theta})$ satisfies assertions (i)–(iv) of the proposition and $\exp : N + N' \to \exp(N + N')$ is a diffeomorphism with inverse given by the log function. Since $N \subset \mathfrak{g}_0$ and

$$\theta = \log(e^{N'} \cdot e^{N}) \supset \log(e^{N}) = N\,,$$

we see that $N \subset \mathfrak{g}_0 \cap \theta$. As $\exp(\mathfrak{g}_0 \cap \theta)$ is a τ_0-connected subset of $G_0 \cap \tilde{\theta}$ which contains I, we must have

$$\exp(\mathfrak{g}_0 \cap \theta) \subset \mathcal{C} = \exp(N)\,. \tag{3.6}$$

Since $\exp : \theta \to \tilde{\theta}$ is bijective and $N \subset \mathfrak{g}_0 \cap \theta \subset \theta$, we may conclude from Eq. (3.6) that $\mathfrak{g}_0 \cap \theta \subset N$. Thus we have shown $\mathfrak{g}_0 \cap \theta = N$ and hence $\exp(\mathfrak{g}_0 \cap \theta) = \exp(N) = \mathcal{C}$, which completes the proof of item (v). $\square$

Definition 3.2. For the rest of this section, we will assume θ has been chosen as in Proposition 3.1. Moreover, we will let $U := \theta \cap \mathfrak{g}_0$ and $\tilde{U} := \exp(U)$ which may also be described as the τ_0-connected component of $G_0 \cap \tilde{\theta}$ which contains $I \in G_0$.

Since $\mathfrak{g}_0 \subset \mathrm{gl}(n,\mathbb{R})$, $\mathfrak{g}_0$ inherits the Hilbert–Schmidt inner product from $\mathrm{gl}(n,\mathbb{R})$ and this inner product induces a unique right-invariant Riemannian metric on G_0. Let d_{G_0} denote the induced right-invariant distance on G_0. That is, for x, $y \in G_0$,

$$d_{G_0}(x,y) = \inf_\sigma \int_0^1 |\sigma'(s)\sigma^{-1}(s)|ds\,, \tag{3.7}$$

where the infimum is now taken over all piecewise C^1 paths, $\sigma : [0,1] \to G_0$ with $\sigma(0) = x$ and $\sigma(1) = y$.

Lemma 3.2. *Suppose $\gamma : [0,T] \to \mathrm{GL}(n,\mathbb{R})$ is a continuous map such that $\gamma(t) \in G_0$ for all $t \in [0,T]$. Then γ is a τ_0-continuous map into G_0. Moreover, if we further assume that $\gamma([0,T]) \subset G_0 \cap \tilde{\theta}$, then*

(a) *$\gamma(t) \in \tilde{U}$ for all $t \in [0,T]$, and*
(b) *$d_{G_0}(I,\gamma(t)) \le C|I - \gamma(t)|$ for all $t \in [0,T]$, where C is the constant appearing in Eq.* (3.5).

Proof. The assertion that γ is continuous as a map into G_0 follows from Theorem 1.62 of Ref. 26 and the construction of G_0 as a maximal integral manifold (Theorem 3.19 of Ref. 26) as described above.

For assertion (a), notice that $\gamma([0,T])$ is the τ_0-continuous image of a connected set and is therefore a τ_0-connected subset of $G_0 \cap \tilde{\theta}$. As $I \in \gamma([0,T])$, it now follows that $\gamma([0,T]) \subset \tilde{U}$.

Finally, for any Lie group, $d(I, e^A) \le |A|$ which follows from the fact that $s \to e^{sA}$ is a path joining I, at $s = 0$, to e^A, at $s = 1$ and hence

$$d(I,e^A) \le \int_0^1 \left|\left(\frac{d}{ds}e^{sA}\right)e^{-sA}\right| ds = \int_0^1 |A|ds\,.$$

Assertion (b) follows by substituting $A = \log\gamma(t)$ and using Eq. (3.5). $\square$

Lemma 3.3. *Let $\rho : H \to G$ be a Lie homomorphism of two Lie groups, H and G, $\mathfrak{h} := \mathrm{Lie}(H)$, $\mathfrak{g} := \mathrm{Lie}(G)$, Γ be a finite subset of $\mathfrak{h}$, and $\{B^A\}_{A\in\Gamma}$ be a collection of independent Brownian motions. If $h_t \in H$ solves the stochastic differential equation,*

$$\delta h_t = \sum_{A\in\Gamma} \tilde{A}(h_t)\delta B_t^A \text{ with } h_0 = e \in H\,, \tag{3.8}$$

then $g_t := \rho(h_t) \in G$ solves the stochastic differential equation,

$$\delta g_t = \sum_{A\in\Gamma} \widetilde{\rho_* A}(g_t)\delta B_t^A \text{ with } g_0 = e \in G\,, \tag{3.9}$$

where $\rho_* A := \frac{d}{dt}|_0 \rho(e^{tA}) \in \mathfrak{g}$.

Proof. For $f \in C^\infty(G)$ (so that $f \circ \rho \in C^\infty(H)$) and $A \in \mathfrak{g}$ we have

$$\begin{aligned}\tilde{A}(f\circ\rho)(h) &= \frac{d}{dt}|_0 f(\rho(he^{tA})) = \frac{d}{dt}|_0 f(\rho(h)\rho(e^{tA})) \\ &= (\widetilde{\rho_* A} f)(\rho(h))\,.\end{aligned}$$

Therefore it follows that

$$\delta f(g_t) = \delta(f \circ \rho(h_t)) = \sum_{A \in \Gamma} (\tilde{A}(f \circ \rho))(h_t)\delta B_t^A$$
$$= \sum_{A \in \Gamma} (\widetilde{\rho_* A} f)(\rho(h_t))\delta B_t^A = \sum_{A \in \Gamma} (\widetilde{\rho_* A} f)(g_t)\delta B_t^A .$$

That this last identity holds for all $f \in C^\infty(G)$ is precisely the meaning of the first identity in Eq. (3.9). This completes the proof since $g_0 = \rho(h_0) = \rho(e) = e \in G$. $\square$

Proposition 3.2. *Let* $\tilde{\Gamma}$ *be any linearly independent subset of* $\mathfrak{g}_0$, $B_t = \sum_{A \in \tilde{\Gamma}} B^A(t)A$ *where* $\{B^A\}_{A \in \tilde{\Gamma}}$ *are independent Brownian motions, and let* $g_t \in \mathrm{GL}(n, \mathbb{R})$ *solve the stochastic differential equation*

$$\delta g_t = g_t \delta B_t \text{ with } g_0 = I \in \mathrm{GL}(n, \mathbb{R}) . \tag{3.10}$$

Then, almost surely, $g_t \in G_0$ *for all* t *and* $g. = h.$ *where*

$$\delta h_t = L_{h_t *} \delta B_t \text{ with } h_0 = I \in G_0 . \tag{3.11}$$

The stochastic differential equation in Eq. (3.11) *is to be interpreted as an equation on* G_0.

Proof. By Theorem 4.8.7 of Ref. 19, there exists a unique solution, $h_t \in G_0$, to the stochastic differential equation in Eq. (3.11). Applying Lemma 3.3 with $H = G_0$, $G = \mathrm{GL}(n, \mathbb{R})$, and ρ being the inclusion map, shows $\rho(h_t) = h_t$ also solves Eq. (3.10) with g changed to h. Since solutions to Eq. (3.10) are unique, we know that $g. = h.$ a.s. $\square$

We now prove Theorem 1.1 with G replaced by G_0. We will let $\beta_0(t, s) := \varphi(\beta(t, s))$, where $\beta(t, s)$ is given in Eq. (1.1) and $\varphi : \mathfrak{g} \to \mathfrak{g}_0 \subset \mathrm{gl}(n, \mathbb{R})$ is a faithful representation of $\mathfrak{g}$.

Theorem 3.1. *For each* $s \in [0, 1]$, *let* $g(t, s)$ *be the* G_0 *valued process solving*

$$g(dt, s) = g(t, s)\beta_0(\delta t, s) \text{ with } g(0, s) = I . \tag{3.12}$$

Then $g(t, s)$ *has a version,* $g_0(t, s)$, *which is jointly continuous in the manifold topology,* τ_0, *on* G_0.

Remark 3.2. Proposition 3.2 gives that g in Eq. (3.12) is G_0-valued, and Theorem 2.1 indicates that it has a jointly continuous version when viewed as a $\mathrm{GL}(n, \mathbb{R})$-valued process. It remains to show how this implies a jointly continuous version in the τ_0-topology on G_0.

Proof. Let $\Lambda := \cup_{n=1}^{\infty} \{m2^{-n} | 0 \leq m \leq 2^n\}$ be the dyadic rationals in [0, 1]. Applying Kolmogorov's continuity criterion (Theorem 1.4.1 of Ref. 19) to Eq. (2.40) implies that, for every $\alpha \in (0, 1/2)$, there exists a random variable $C_\alpha : \Omega \to (0, \infty]$ such that $C_\alpha < \infty$ a.s. and

$$\sup_{0 \le t \le T} |g(t, s) - g(t, \sigma)| \le C_\alpha |s - \sigma|^\alpha \ \forall \ \sigma, s \in \Lambda\,. \tag{3.13}$$

By Proposition 2.1, $\tilde{C} := \sup_{0 \le t \le T} |g(t, \sigma)^{-1}| < \infty$ a.s. Since

$$\begin{aligned} |g(t, s)g(t, \sigma)^{-1} - I| &= |(g(t, s) - g(t, \sigma))g(t, \sigma)^{-1}| \\ &\le |g(t, s) - g(t, \sigma)||g(t, \sigma)^{-1}|\,, \end{aligned}$$

we have

$$\sup_{0 \le t \le T} |g(t, s)g(t, \sigma)^{-1} - I| \le \tilde{C}_\alpha |s - \sigma|^\alpha \ \forall \ \sigma\,, \qquad s \in \Lambda\,, \tag{3.14}$$

where $\tilde{C}_\alpha := \tilde{C} C_\alpha < \infty$ a.s.

Let $\delta > 0$ be chosen so that if $x \in \mathrm{GL}(n, \mathbb{R})$ satisfies, $|I - x| < \delta$, then $x \in \tilde{\theta}$ where $\tilde{\theta}$ is as in Proposition 3.1. As $\tilde{C}_\alpha < \infty$ a.s., the random variable, $\varepsilon := \delta / \tilde{C}_\alpha$, is almost surely positive. Hence at a sample point, $\omega \in \Omega$, where $\tilde{C}_\alpha(\omega) < \infty$ (equivalently $\varepsilon(\omega) > 0$), we have $(t \to g(t, s)g(t, \sigma)^{-1}) \in C([0, T], \tilde{\theta})$ provided σ, $s \in \Lambda$ and $|s - \sigma| < \varepsilon$. Since $g(t, s)g(t, \sigma)^{-1} \in G_0$ as well, we may apply Lemma 3.2 to conclude that the map, $(t \to g(t, s)g(t, \sigma)^{-1}) : [0, T] \to G_0 \cap \tilde{\theta}$, is τ_0-continuous. Moreover for s, $\sigma \in \Lambda$ with $|s - \sigma| < \varepsilon$, Lemma 3.2 implies that

$$\sup_{0 \le t \le T} d_G(g(t, s)g(t, \sigma)^{-1}, I) \le C \sup_{0 \le t \le T} |g(t, s)g(t, \sigma)^{-1} - I| \le C\tilde{C}_\alpha |s - \sigma|^\alpha\,,$$

wherein the last inequality we have used Eq. (3.14). Since our metric was chosen to be right invariant, this equation may be written as,

$$\sup_{0 \le t \le T} d_{G_0}(g(t, s), g(t, \sigma)) \le C\tilde{C}_\alpha |s - \sigma|^\alpha\,,$$

provided that s, $\sigma \in \Lambda$ and $|s - \sigma| < \varepsilon$. By repeated use of the triangle inequality we may now conclude that

$$\sup_{0 \le t \le T} d_{G_0}(g(t, s), g(t, \sigma)) \le \bar{C}_\alpha |s - \sigma|^\alpha \ \forall \ \sigma\,, \qquad s \in \Lambda\,,$$

where

$$\bar{C}_\alpha := C\tilde{C}_\alpha(1 + \varepsilon^{-1}) = C\tilde{C}_\alpha(1 + \delta^{-1}\tilde{C}_\alpha) < \infty \text{ a.s.}$$

This shows that (almost surely) $\Lambda \ni s \to g(\cdot, s)$ is an α-Hölder continuous map into $C([0, T] \to G_0)$, where $C([0, T] \to G_0)$ is equipped with the uniform metric associated to d_{G_0}. This map is therefore uniformly continuous and extends uniquely to a Hölder continuous map from $[0, 1] \to C([0, T] \to G_0)$ which is the desired version. That is, if we define

$$g_0(t, s) := \lim_{\substack{\lambda \in \Lambda \\ \lambda \to s}} g(t, \lambda)\,,$$

then since $[0, 1] \ni s \to g(\cdot, s) \in L^p(\Omega, C([0, T] \to \mathrm{gl}(n, \mathbb{R}))$ is continuous (see Eq. (2.40)), we may conclude that $g_0(\cdot, s) = g(\cdot, s)$ a.s. for all $s \in [0, 1]$. So that g_0 is indeed a version of g. ◻

4. Pushing Down and Lifting Up

Up to this point we have constructed a jointly continuous G_0-valued process satisfying the desired stochastic differential equation, Eq. (3.12). We have yet to indicate how this implies Theorem 1.1, since G is not a matrix group in general. This will be accomplished by first lifting the process, $g_0(t,s) \in G_0$, of Theorem 3.1 to the universal cover, $\tilde{G}$, of both G and G_0 and then pushing the resulting process down to G. The next proposition explains the "pushing down" procedure in this covering group context.

Lemma 4.1. (Pushing down) *Suppose $\rho : \tilde{H} \to H$ is a Lie group homomorphism which is also a covering map. Let $\tilde{\mathfrak{h}} := \mathrm{Lie}(\tilde{H})$ and $\mathfrak{h} = \mathrm{Lie}(H)$, so that $\rho_{\tilde{e}*} : \tilde{\mathfrak{h}} \to \mathfrak{h}$ is a Lie algebra isomorphism, where $\tilde{e} \in \tilde{H}$ is the identity. Let Γ be any linearly independent subset of $\mathfrak{h}$,*

$$\tilde{\Gamma} := \rho_{\tilde{e}*}^{-1}(\Gamma) := \{\underline{A} := \rho_{\tilde{e}*}^{-1} A \in \tilde{\mathfrak{h}} : A \in \Gamma\} \subset \tilde{\mathfrak{h}}\,,$$

$\beta(t,s) = \sum_{A\in\Gamma} \beta^A(t,s)A$, and $\tilde{\beta}(t,s) = \sum_{A\in\Gamma} \beta^A(t,s)\underline{A}$, where $\{\beta^A\}_{A\in\Gamma}$ are independent Brownian sheets or Brownian bridge sheets. If $\tilde{g}(t,s) \in \tilde{H}$ is a process satisfying, for each $s \in [0,1]$,

$$\tilde{g}(\delta t,s) = L_{\tilde{g}(t,s)*}\tilde{\beta}(\delta t,s) \ \text{with} \ \tilde{g}(0,s) = \tilde{e} \in \tilde{H} \tag{4.1}$$

then $g(t,s) := \rho(\tilde{g}(t,s))$ is an H-valued process satisfying,

$$g(\delta t,s) = L_{g(t,s)*}\beta(\delta t,s) \ \text{with} \ g(0,s) = e \in H\,. \tag{4.2}$$

Proof. This is a special case of Lemma 3.3 with $\beta(t,s)$ being decomposed as in Eq. (2.22). Alternatively, one can simply repeat the proof of Lemma 3.3 in this context. □

Let $\tilde{G}$ be the unique (up to isomorphism) simply connected Lie group such that $\tilde{\mathfrak{g}} := \mathrm{Lie}(\tilde{G})$ is isomorphic to $\mathfrak{g}$. The group $\tilde{G}$ is the universal cover for any Lie group whose Lie algebra is isomorphic to $\mathfrak{g}$. Hence there exist covering maps, $\rho : \tilde{G} \to G$ and $\rho_0 : \tilde{G} \to G_0$. The maps ρ and ρ_0 are also Lie group homomorphisms which are also locally isomorphisms. By constructing $\rho_0 : \tilde{G} \to G$ to be the unique Lie group homomorphism such that $\rho_{0*\tilde{e}} = \varphi\rho_{*\tilde{e}}$, we may further assume that $\rho_{0*\tilde{e}} = \varphi\rho_{*\tilde{e}}$.

Since $[0,\infty) \times [0,1]$ is contractible, the continuous G_0-valued process, $g_0(t,s)$, of Theorem 3.1 has a unique lift to a continuous process $\tilde{G}$-valued process, $\tilde{g}(t,s)$, see for example Lemma 14.2 of Ref. 23. More explicitly, $\tilde{g}(t,s)$ is the the unique jointly continuous process in $\tilde{G}$ such that $\tilde{g}(0,0) = e$ and $\rho_0(\tilde{g}(t,s)) = g_0(t,s)$. Since $g_0(t,0) = I$ for all t, and $\tilde{g}(\cdot,0)$ is a lift of $g_0(\cdot,0)$, we may further conclude that $\tilde{g}(t,0) = \tilde{e} \in \tilde{G}$ for all t.

Corollary 4.1. (Lifting up) *The continuous $\tilde{G}$-valued process, $\tilde{g}(t,s)$, described above satisfies the stochastic differential equation,*

$$\tilde{g}(\delta t,s) = L_{\tilde{g}(t,s)*}\tilde{\beta}(\delta t,s) \ \text{with} \ \tilde{g}(0,s) = \tilde{e} \in \tilde{G}\,, \tag{4.3}$$

where $\tilde{\beta}(t,s) := \rho_{*\tilde{e}}^{-1}\beta(t,s) = \rho_{0*\tilde{e}}^{-1}\beta_0(t,s)$, *with* $\beta(t,s)$ *given as in Eq.* (1.1) *and* $\beta_0(t,s)$ *as in Theorem* 3.1.

Proof. Let $g_0(t,s)$ be as in Theorem 3.1. Fix an $s \in [0,1]$ and let $\bar{g}(\cdot,s)$ solve (see Theorem 4.8.7 of Ref. 19),

$$\bar{g}(\delta t,s) = L_{\bar{g}(t,s)*}\tilde{\beta}(\delta t,s) \text{ with } \bar{g}(0,s) = \tilde{e} \in \tilde{G}\,. \tag{4.4}$$

By the push down Lemma 4.1, $\rho_0(\bar{g}(\cdot,s))$ solves the same stochastic differential equation as $g_0(\cdot,s)$ and therefore by uniqueness of such solutions, we know that $\rho_0(\bar{g}(\cdot,s)) = g_0(\cdot,s)$ a.s. This shows, almost surely, that $\bar{g}(\cdot,s)$ is a lift of $g_0(\cdot,s)$ and since lifts are unique we may conclude that $\bar{g}(\cdot,s) = \tilde{g}(\cdot,s)$ a.s. In particular, $\tilde{g}(\cdot,s)$ solves the same stochastic differential equation as $\bar{g}(\cdot,s)$, i.e. $t \to \tilde{g}(t,s)$ solves Eq. (4.3). $\square$

4.1. *Completion of the proof of Theorem* 1.1

Let $g(t,s) := \rho(\tilde{g}(t,s)) \in G$ where $\tilde{g}(t,s)$ is the jointly continuous $\tilde{G}$-valued process in Corollary 4.1. Clearly $g(t,s)$ is a jointly continuous process which by the push down Lemma 4.1 satisfies, Eq. (1.6) with $\sigma_0(s) \equiv e \in G$. For general $\sigma_0 \in \mathcal{W}(G)$, the process, $\Sigma(t,s) := \sigma_0(s)g(t,s)$, satisfies the conclusions of Theorem 1.1.

5. Heat Equations

We refer to the reader to the Introduction for the definition of cylinder functions and their derivatives. In this section, we will reference results found primarily in Refs. 6, 8 and 9.

Definition 5.1. For $n \in \mathbb{N}$, $i \in \{1,2,\ldots,n\}$, $F \in C^\infty(G^n)$, and $A \in \mathfrak{g}$, let

$$\tilde{A}^{(i)}F(x_1,x_2,\ldots,x_n) := \frac{d}{dt}|_0 F(x_1,\ldots,x_i \cdot e^{tA}, x_{i+1},\ldots,x_n) \tag{5.1}$$

for all $(x_1,x_2,\ldots,x_n) \in G^n$.

Remark 5.1. Note that if $f = F \circ \pi_{\mathcal{P}}$ as in Definition 1.2, then

$$\tilde{h}f = \sum_{i=1}^{n} (\widetilde{h(s_i)}^{(i)} F) \circ \pi_{\mathcal{P}} \ \forall\, h \in H(V)\,. \tag{5.2}$$

In particular, note that $\tilde{h}f$ is still a smooth cylinder function based on the same partition $\mathcal{P}$.

Recalling the definitions of k, $k_0 : [0,1]^2 \to [0,1]$ in Eq. (1.2), we have, by the same methods used to prove Lemma 3.8 in Ref. 8 or Lemma 3.3 in Ref. 9, that

$$\sum_{h\in S} h(\sigma) \otimes h(s) = k(s,t) \sum_{A\in\Gamma} A \otimes A \text{ and}$$

$$\sum_{h\in S_0} h(\sigma) \otimes h(s) = k_0(s,t) \sum_{A\in\Gamma} A \otimes A\,.$$

From these identities and Eq. (5.2) it easily follows, see Remark 3.5 in Ref. 9, that if $f = F \circ \pi_{\mathcal{P}}$, then

$$L_{H(V)}(F \circ \pi_{\mathcal{P}}) = (L_{\mathcal{P}} F) \circ \pi_{\mathcal{P}}$$

and

$$L_{H_0(V)} f = (L^0_{\mathcal{P}} F) \circ \pi_{\mathcal{P}} \,,$$

where $L_{\mathcal{P}}$ and $L^0_{\mathcal{P}}$ are the operators on $C^\infty(G^{\#(\mathcal{P})})$ defined by,

$$L_{\mathcal{P}} F := \sum_{A \in \Gamma} \sum_{i,j=1}^{n} k(s_j, s_i)(\tilde{A}^{(j)} \tilde{A}^{(i)} F) \,, \tag{5.3}$$

and

$$L^0_{\mathcal{P}} F := \sum_{A \in \Gamma} \sum_{i,j=1}^{n} k_0(s_j, s_i)(\tilde{A}^{(j)} \tilde{A}^{(i)} F) \,. \tag{5.4}$$

Definition 5.2. Given the Brownian sheet or Brownian bridge sheet, $\beta(t, s)$ (see Eq. (1.1)), and a partition, $\mathcal{P}$, of [0, 1] as in Eq. (1.4), let

$$\beta_{\mathcal{P}}(t) := (\beta(t, s_1), \beta(t, s_2), \ldots, \beta(t, s_n)) \,.$$

In the next proposition, $\Sigma(t, s)$ will denote the continuous G-valued process described in Theorem 1.1 with $\sigma_0(s) \equiv e \in G$ for all $s \in [0, 1]$. When β is the Brownian bridge sheet, we will denote Σ by Σ^0.

Proposition 5.1. *If $\beta(t, s)$ is a Brownian sheet, $\mathcal{P}$ is a partition of $[0, 1]$, and $\Sigma_{\mathcal{P}}(t) := \pi_{\mathcal{P}} \circ \Sigma(t, \cdot)$, then $\Sigma_{\mathcal{P}}$ solves the stochastic differential equation,*

$$\Sigma_{\mathcal{P}}(\delta t) = L_{\Sigma_{\mathcal{P}}(t)*} \beta_{\mathcal{P}}(\delta t) \text{ with } \Sigma_{\mathcal{P}}(0) = (e, e, \ldots, e) \in G^{\#(\mathcal{P})} \,,$$

and has $\frac{1}{2} L_{\mathcal{P}}$ (see Eq. (5.3)) as its generator. Similarly, if $\beta(t, s)$ is a Brownian bridge sheet and $\Sigma^0_{\mathcal{P}}(t) := \pi_{\mathcal{P}} \circ \Sigma^0(t, \cdot)$, then $\Sigma^0_{\mathcal{P}}$ solves

$$\Sigma^0_{\mathcal{P}}(\delta t) = L_{\Sigma^0_{\mathcal{P}}(t)*} \beta_{\mathcal{P}}(\delta t) \text{ with } \Sigma^0_{\mathcal{P}}(0) = (e, e, \ldots, e) \in G^{\#(\mathcal{P})}$$

and has $\frac{1}{2} L^0_{\mathcal{P}}$ (see Eq. (5.4)) as its generator.

Proof. We refer the reader to Sec. 3.3 of Ref. 6, specifically the proof of Theorem 3.10, for a proof of the second statement. This proof is valid despite the difference in starting assumptions (in Ref. 6, the group G was assumed to be compact and Γ was assumed to span $\mathfrak{g}$). The proof of the first statement follows by replacing k_0 with k. $\square$

5.1. *Proof of Proposition* 1.1

If $f = F \circ \pi_{\mathcal{P}}$, then by Proposition 5.1 and Eqs. (5.3) and (5.4),

$$\begin{aligned}\frac{\partial}{\partial t}\nu_t(f) &= \frac{\partial}{\partial t}\mathbb{E}[F(\Sigma_{\mathcal{P}}(t)] = \frac{1}{2}\mathbb{E}[(L_{\mathcal{P}}F)(\Sigma_{\mathcal{P}}(t)] \\ &= \frac{1}{2}\mathbb{E}[(L_{H(V)}f)(\Sigma(t))] = \frac{1}{2}\nu_t(L_{H(V)}f)\end{aligned}$$

and

$$\lim_{t\downarrow 0}\nu_t(f) = \lim_{t\downarrow 0}\mathbb{E}[F(\Sigma_{\mathcal{P}}(t))] = F(e,e,\ldots,e) = f(\mathbf{e})\,.$$

This proves Eq. (1.12). Equation (1.13) is proved analogously.

Appendix

Suppose that $\Gamma \subset \mathfrak{g}$ is a finite subset which is not necessarily independent in $\mathfrak{g}$. We again may let

$$\beta(t,s) := \sum_{A\in\Gamma}\beta^A(t,s)A\,, \tag{A.1}$$

where $\{\beta^A\}_{A\in\Gamma}$ are independent mean zero Gaussian fields with covariances described in Eq. (1.1). The following lemma shows that there is no loss of generality, for the purposes of this paper, in assuming that Γ is in fact a linearly independent subset of $\mathfrak{g}$.

Lemma A.1. *Let $V := \mathrm{span}(\Gamma) \subset \mathfrak{g}$. There is a unique inner product on, $(\cdot,\cdot)_V$, on V such that if $\{A_i\}_{i=1}^{\dim(V)}$ is an orthonormal basis for V, then*

$$\beta(t,s) = \sum_{i=1}^{\dim(V)}\beta^i(t,s)A_i\,,$$

where $\{\beta^i(s,t) := (\beta(t,s),A_i)\}_{i=1}^{\dim(V)}$ is again a collection of independent $\mathbb{R}$-valued Brownian sheets or Brownian bridge sheets.

Proof. For α, $\beta \in V^*$, let $q(\alpha,\beta) := \sum_{A\in\Gamma}\alpha(A)\beta(A)$. Then q defines an inner product on V^* which in turn induces an inner product, $(\cdot,\cdot)_V$, on V such that $\{A_i\}_{i=1}^{\dim(V)} \subset V$ is an orthonormal basis for $(\cdot,\cdot)_V$ iff the the dual basis, $\{\alpha^i\}_{i=1}^{\dim(V)}$, is an orthonormal basis for $(V^*, q(\cdot,\cdot))$. For such a basis we have

$$\beta(t,s) = \sum_{i=1}^{\dim(V)}\alpha^i(\beta(t,s))A_i\,,$$

where $\alpha^i(\beta(t,s))$ are mean zero Gaussian processes such that

$$\begin{aligned}
\mathbb{E}[\alpha^i(\beta(t,s))\alpha^j(\beta(t,s))] &= \sum_{A,B\in\Gamma} \mathbb{E}[\beta^A(t,s)\beta^B(t,s)][\alpha^i(A)\alpha^j(B)] \\
&= \bar{k}(\sigma,s)(t\wedge\tau)\sum_{A\in\Gamma}[\alpha^i(A)\alpha^j(A)] \\
&= \bar{k}(\sigma,s)(t\wedge\tau)q(\alpha^i,\alpha^j) = \delta_{ij}k(\sigma,s)(t\wedge\tau)\,.
\end{aligned}$$

As $\alpha^i(\beta(t,s)) = (\beta(t,s),A_i)$, the proof is complete. □

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