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Heat kernel analysis on infinite-dimensional Heisenberg groups

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Abstract

We introduce a class of non-commutative Heisenberg-like infinite-dimensional Lie groups based on an abstract Wiener space. The Ricci curvature tensor for these groups is computed and shown to be bounded. Brownian motion and the corresponding heat kernel measures, $\{\nu_t\}_{t>0}$, are also studied. We show that these heat kernel measures admit: (1) Gaussian like upper bounds, (2) Cameron–Martin type quasi-invariance results, (3) good L^p-bounds on the corresponding Radon–Nikodym derivatives, (4) integration by parts formulas, and (5) logarithmic Sobolev inequalities. The last three results heavily rely on the boundedness of the Ricci tensor.

Keywords: Heisenberg group; Heat kernel; Quasi-invariance; Logarithmic Sobolev inequality

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1. Introduction

Both authors have been greatly influenced by Professor Malliavin and his work over the years. In particular this paper is partially an attempt to better understand Malliavin's paper [40]. It is with great pleasure to us that this article appears in this special edition of JFA dedicated to Professor Paul Malliavin.

The aim of this paper is to construct and study properties of heat kernel measures on certain infinite-dimensional Heisenberg groups. In this paper the Heisenberg groups will be constructed from a skew symmetric form on an abstract Wiener space. A typical example of such a group is the Heisenberg group of a symplectic vector space. Before describing our results let us recall some heat kernel results for finite-dimensional Riemannian manifolds.

1.1. A finite-dimensional paradigm

Let (M, g) be a complete connected n-dimensional Riemannian manifold ($n < \infty$), $\Delta = \Delta_g$ be the Laplace–Beltrami operator acting on $C^2(M)$, and Ric denote the associated Ricci tensor. Recall (see for example Strichartz [48], Dodziuk [15] and Davies [12]) that the closure, $\bar{\Delta}$, of $\Delta|_{C_c^\infty(M)}$ is self-adjoint on $L^2(M, dV)$, where $dV = \sqrt{g}\, dx^1 \dots dx^n$ is the Riemann volume measure on M. Moreover, the semi-group $P_t := e^{t\bar{\Delta}/2}$ has a symmetric positive integral (heat) kernel, $p_t(x, y)$, such that $\int_M p_t(x, y)\, dV(y) \leqslant 1$ for all $x \in M$ and

$$P_t f(x) := \left(e^{t\bar{\Delta}/2} f\right)(x) = \int_M p_t(x, y) f(y)\, dV(y) \quad \text{for all } f \in L^2(M). \tag{1.1}$$

Theorem 1.2 summarizes some of the results that we would like to extend to our infinite-dimensional Heisenberg group setting.

Notation 1.1. If μ is a probability measure on a measure space $(\Omega, \mathcal{F})$ and $f \in L^1(\mu) = L^1(\Omega, \mathcal{F}, \mu)$, we will often write $\mu(f)$ for the integral, $\int_\Omega f\,d\mu$.

Theorem 1.2. *Beyond the assumptions above, let us further assume that* Ric $\geqslant kI$ *for some* $k \in \mathbb{R}$. *Then*

(1) $p_t(x, y)$ *is a smooth function.* (*The Ricci curvature assumption is not needed here.*)
(2) $\int_M p_t(x, y)\,dV(y) = 1$ (*see for example Davies* [12, *Theorem* 5.2.6]).
(3) *Given a point* $o \in M$, *let* $d\nu_t(x) := p_t(o, x)\,dV(x)$ *for all* $t > 0$. *Then* $\{\nu_t\}_{t>0}$ *may be characterized as the unique family of probability measures such that the function* $t \to \nu_t(f) := \int_M f\,d\nu_t$ *is continuously differentiable,*

$$\frac{d}{dt}\nu_t(f) = \frac{1}{2}\nu_t(\Delta f), \quad \text{and} \quad \lim_{t\downarrow 0}\nu_t(f) = f(o) \tag{1.2}$$

for all $f \in BC^2(M)$, *the bounded* C^2*-functions on* M.
(4) *There exist constants,* $c = c(K, n, T)$ *and* $C = C(K, n, T)$, *such that,*

$$p(t, x, y) \leqslant \frac{C}{V(x, \sqrt{t/2})}\exp\left(-c\frac{d^2(x, y)}{t}\right), \tag{1.3}$$

for all $x, y \in M$ *and* $t \in (0, T]$, *where* $d(x, y)$ *is the Riemannian distance from* x *to* y *and* $V(x, r)$ *is the volume of the* r*-ball centered at* x.
(5) *The* heat kernel measure, ν_T, *for any* $T > 0$ *satisfies the following logarithmic Sobolev inequality*:

$$\nu_T\left(f^2 \log f^2\right) \leqslant 2k^{-1}\left(1 - e^{-kT}\right)\nu_T\left(|\nabla f|^2\right) + \nu_T\left(f^2\right)\log \nu_T\left(f^2\right), \tag{1.4}$$

for $f \in C_c^\infty(M)$.

These results are fairly standard. For item (3) see [17, Theorem 2.6], for Eq. (1.3) see for example Theorems 5.6.4, 5.6.6, and 5.4.12 in Saloff-Coste [47] and for more detailed bounds see [12,13,26,39,46]. The logarithmic Sobolev inequality in Eq. (1.4) generalizes Gross' [28] original logarithmic Sobolev inequality valid for $M = \mathbb{R}^n$ and is due in this generality to D. Bakry and M. Ledoux, see [2,3,38]. Also see [10,21,30,51,52] and Driver and Lohrenz [22, Theorem 2.9] for the case of interest here, namely when M is a uni-modular Lie group with a left-invariant Riemannian metric.

When passing to infinite-dimensional Riemannian manifolds we will no longer have available the Riemannian volume measure. Because of this problem, we will take item (3) of Theorem 1.2 as our definition of the heat kernel measure. The heat kernel upper bound in Eq. (1.3) also does not make sense in infinite dimensions. However, the following consequence almost does: there exists $c(T) > 0$ such that

$$\int_M \exp\left(c(T)\frac{d^2(o, x)}{t}\right)d\nu_t(x) < \infty \quad \text{for all } 0 < t \leqslant T. \tag{1.5}$$

In fact Eq. (1.5) will not hold in infinite dimensions either. It will be necessary to replace the distance function, d, by a weaker distance function as happens in Fernique's theorem for Gaussian measure spaces. With these results as background we are now ready to summarize the results of this paper.

1.2. Summary of results

Let us describe the setting informally, for precise definitions see Sections 2 and 3. Let (W, H, μ) be an abstract Wiener space, $\mathbf{C}$ be a finite-dimensional inner product space, and $\omega: W \times W \to \mathbf{C}$ be a continuous skew symmetric bilinear quadratic form on W. The set $\mathfrak{g} = W \times \mathbf{C}$ can be equipped with a Lie bracket by setting

$$\big[(A,a),(B,b)\big] = \big(0, \omega(A,B)\big).$$

As in the case for the Heisenberg group of a symplectic vector space, the Lie algebra $\mathfrak{g} = W \times \mathbf{C}$ can be given the group structure by defining

$$(w_1, c_1) \cdot (w_2, c_2) = \left(w_1 + w_2, c_1 + c_2 + \frac{1}{2}\omega(\omega_1, w_2) \right).$$

The set $W \times \mathbf{C}$ with the group structure will be denoted by G or $G(\omega)$. The Lie subalgebra $\mathfrak{g}_{\mathrm{CM}} = H \times \mathbf{C}$ is called the Cameron–Martin subalgebra, and $\mathfrak{g}_{\mathrm{CM}}$ equipped with the same group multiplication denoted by G_{CM} and called the Cameron–Martin subgroup. We equip G_{CM} with the left-invariant Riemannian metric which agrees with the natural Hilbert inner product,

$$\big\langle (A,a),(B,b)\big\rangle_{\mathfrak{g}_{\mathrm{CM}}} := \langle A, B\rangle_H + \langle a, b\rangle_{\mathbf{C}},$$

on $\mathfrak{g}_{\mathrm{CM}} \cong T_{\mathbf{e}} G_{\mathrm{CM}}$. In Section 3 we give several examples of this abstract setting including the standard finite-dimensional Heisenberg group.

The main objects of our study are a Brownian motion in G and the corresponding heat kernel measure defined in Section 4. Namely, let $\{(B(t), B_0(t))\}_{t \geqslant 0}$ be a Brownian motion on $\mathfrak{g}$ with variance determined by

$$\begin{aligned} &\mathbb{E}\big[\big\langle \big(B(s), B_0(s)\big), (A,a)\big\rangle_{\mathfrak{g}_{\mathrm{CM}}} \cdot \big\langle \big(B(t), B_0(t)\big), (C,c)\big\rangle_{\mathfrak{g}_{\mathrm{CM}}}\big] \\ &\quad = \mathrm{Re}\big\langle (A,a),(C,c)\big\rangle_{\mathfrak{g}_{\mathrm{CM}}} \min(s,t) \end{aligned}$$

for all $s, t \in [0,\infty)$, $A, C \in H_*$ and $a, c \in \mathbf{C}$. Then the Brownian motion on G is the continuous G-valued process defined by

$$g(t) = \left(B(t), B_0(t) + \frac{1}{2}\int_0^t \omega\big(B(\tau), dB(\tau)\big) \right).$$

For $T > 0$ the heat kernel measure on G is $\nu_T = \mathrm{Law}(g(T))$. It is shown in Corollary 4.5 that $\{\nu_t\}_{t>0}$ satisfies item (3) of Theorem 1.2 with $o = (0,0) \in G(\omega)$.

Theorem 4.16 gives heat kernel measure bounds that may be viewed as a non-commutative version of Fernique's theorem for $G(\omega)$. In light of Theorem 3.12 this result is also analogous to the integrated Gaussian upper bound in Eq. (1.5).

In Theorem 5.2 we prove quasi-invariance for the path space measure associated to the Brownian motion, g, on G with respect to multiplication on the left by finite energy paths in the Cameron–Martin subgroup G_{CM}. (In light of the results in Malliavin [40] it is surprising that Theorem 5.2 holds.) Theorem 5.2 is then used to prove quasi-invariance of the heat kernel measures with respect to both right and left multiplication (Theorem 6.1 and Corollary 6.2), as well as integration by parts formulae on the path space and for the heat kernel measures, see Corollaries 5.6–6.5. These results can be interpreted as the first steps towards proving ν_t is a "strictly positive" smooth measure. In this infinite-dimensional setting it is natural to interpret quasi-invariance and integration by parts formulae as properties of the smoothness of the heat kernel measure, see [17, Theorem 3.3] for example.

In Section 7 we compute the Ricci curvature and check that not only is it bounded from below (see Proposition 7.2), but also that the Ricci curvature of certain finite-dimensional "approximations" are bounded from below with constants independent of the approximation. Based on results in [18], these bounds allow us to give another proof of the quasi-invariance result for ν_t and at the same time to get L^p-estimates on the corresponding Radon–Nikodym derivatives, see Theorem 8.1. These estimates are crucial for the heat kernel analysis on the spaces of holomorphic functions which is the subject of our paper [19]. In Theorem 8.3 we show that an analogue of the logarithmic Sobolev inequality in Eq. (1.4) holds in our setting as well.

In Section 9 we give a list of open questions and further possible developments of the results of this paper. We expect our methods to be applicable to a much larger class of infinite-dimensional nilpotent groups.

Finally, we refer to papers of H. Airault, P. Malliavin, D. Bell, Y. Inahama concerning quasi-invariance, integration by parts formulae and the logarithmic Sobolev inequality on certain infinite-dimensional curved spaces [1,4–6,32].

2. Abstract Wiener space preliminaries

Suppose that X is a real separable Banach space and $\mathcal{B}_X$ is the Borel σ-algebra on X.

Definition 2.1. A measure μ on $(X, \mathcal{B}_X)$ is called a (mean zero, non-degenerate) *Gaussian measure* provided that its characteristic functional is given by

$$\hat{\mu}(u) := \int_X e^{iu(x)}\, d\mu(x) = e^{-\frac{1}{2}q(u,u)} \quad \text{for all } u \in X^*, \tag{2.1}$$

where $q = q_\mu : X^* \times X^* \to \mathbb{R}$ is a quadratic form such that $q(u,v) = q(v,u)$ and $q(u) = q(u,u) \geqslant 0$ with equality iff $u = 0$, i.e. q is a real inner product on X^*.

In what follows we frequently make use of the fact that

$$C_p := \int_X \|x\|_X^p\, d\mu(x) < \infty \quad \text{for all } 1 \leqslant p < \infty. \tag{2.2}$$

This is a consequence of Skorohod's inequality (see for example [36, Theorem 3.2])

$$\int_X e^{\lambda \|x\|_X}\, d\mu(x) < \infty \quad \text{for all } \lambda < \infty; \tag{2.3}$$

or the even stronger Fernique's inequality (see for example [8, Theorem 2.8.5] or [36, Theorem 3.1])

$$\int_X e^{\delta \|x\|_X^2}\, d\mu(x) < \infty \quad \text{for some } \delta > 0. \tag{2.4}$$

Lemma 2.2. *If $u, v \in X^*$, then*

$$\int_X u(x)v(x)\, d\mu(x) = q(u, v) \tag{2.5}$$

and

$$\left|q(u, v)\right| \leqslant C_2 \|u\|_{X^*} \|v\|_{X^*}. \tag{2.6}$$

Proof. Let $u_*\mu := \mu \circ u^{-1}$ denote the law of u under μ. Then by Eq. (2.1),

$$(u_*\mu)(dx) = \frac{1}{\sqrt{2\pi q(u, u)}} e^{-\frac{1}{2q(u,u)}x^2}\, dx$$

and hence,

$$\int_X u^2(x)\, d\mu(x) = q_\mu(u, u) = q(u, u). \tag{2.7}$$

Polarizing this identity gives Eq. (2.5) which along with Eq. (2.2) implies Eq. (2.6). □

The next theorem summarizes some well-known properties of Gaussian measures that we will use freely below.

Theorem 2.3. *Let μ be a Gaussian measure on a real separable Banach space, X. For $x \in X$ let*

$$\|x\|_H := \sup_{u \in X^* \setminus \{0\}} \frac{|u(x)|}{\sqrt{q(u, u)}} \tag{2.8}$$

and define the Cameron–Martin subspace*, $H \subset X$, by*

$$H = \left\{h \in X\colon\ \|h\|_H < \infty\right\}. \tag{2.9}$$

Then

(1) *H is a dense subspace of X.*
(2) *There exists a unique inner product, $\langle\cdot,\cdot\rangle_H$ on H such that $\|h\|_H^2 = \langle h, h\rangle$ for all $h \in H$. Moreover, with this inner product H is a separable Hilbert space.*
(3) *For any $h \in H$*

$$\|h\|_X \leqslant \sqrt{C_2}\|h\|_H, \tag{2.10}$$

where C_2 is as in (2.2).
(4) *If $\{e_j\}_{j=1}^\infty$ is an orthonormal basis for H, then for any $u, v \in H^*$*

$$q(u,v) = \langle u, v\rangle_{H^*} = \sum_{j=1}^\infty u(e_j)v(e_j). \tag{2.11}$$

The proof of this standard theorem is relegated to Appendix A, see Theorem A.1.

Remark 2.4. It follows from Eq. (2.10) that any $u \in X^*$ restricted to H is in H^*. Therefore we have

$$\int_X u^2\,d\mu = q(u,u) = \|u\|_{H^*}^2 = \sum_{j=1}^\infty \left|u(e_j)\right|^2, \tag{2.12}$$

where $\{e_j\}_{j=1}^\infty$ is an orthonormal basis for H. More generally, if φ is a linear bounded map from X to $\mathbf{C}$, where $\mathbf{C}$ is a real Hilbert space, then

$$\|\varphi\|_{H^*\otimes\mathbf{C}}^2 =: \sum_{j=1}^\infty \left\|\varphi(e_j)\right\|_{\mathbf{C}}^2 = \int_X \left\|\varphi(x)\right\|_{\mathbf{C}}^2\,d\mu(x) < \infty. \tag{2.13}$$

To prove Eq. (2.13), let $\{f_j\}_{j=1}^\infty$ be an orthonormal basis for $\mathbf{C}$. Then

$$\begin{aligned}\int_X \|\varphi(x)\|_{\mathbf{C}}^2\,d\mu(x) &= \int_X \sum_{j=1}^\infty \left|\left\langle\varphi(x), f_j\right\rangle_{\mathbf{C}}\right|^2 d\mu(x) = \sum_{j=1}^\infty \int_X \left|\left\langle\varphi(x), f_j\right\rangle_{\mathbf{C}}\right|^2 d\mu(x)\\ &= \sum_{j=1}^\infty \left\|\left\langle\varphi(\cdot), f_j\right\rangle_{\mathbf{C}}\right\|_{H^*}^2 = \sum_{j=1}^\infty\sum_{k=1}^\infty \left|\left\langle\varphi(e_k), f_j\right\rangle_{\mathbf{C}}\right|^2\\ &= \sum_{k=1}^\infty\sum_{j=1}^\infty \left|\left\langle\varphi(e_k), f_j\right\rangle_{\mathbf{C}}\right|^2 = \sum_{k=1}^\infty \left\|\varphi(e_k)\right\|_{\mathbf{C}}^2 = \|\varphi\|_{H^*\otimes\mathbf{C}}^2.\end{aligned}$$

A simple consequence of Eq. (2.13) is that

$$\|\varphi\|_{H^*\otimes\mathbf{C}}^2 \leqslant \|\varphi\|_{X^*\otimes\mathbf{C}}^2 \int_X \|x\|_X^2\,d\mu(x) = C_2\|\varphi\|_{X^*\otimes\mathbf{C}}^2. \tag{2.14}$$

3. Infinite-dimensional Heisenberg type groups

Throughout the rest of this paper (X, H, μ) will denote a *real abstract Wiener space*, i.e. X is a real separable Banach space, H is a real separable Hilbert space densely embedded into X, and μ is a Gaussian measure on $(X, \mathcal{B}_X)$ such that Eq. (2.1) holds with $q(u,u) := \langle u|_H, u|_H\rangle_{H^*}$.

Following the discussion in [35] and [23] we will say that a (possibly infinite-dimensional) Lie algebra, $\mathfrak{g}$, is of *Heisenberg type* if $\mathbf{C} := [\mathfrak{g}, \mathfrak{g}]$ is contained in the center of $\mathfrak{g}$. If $\mathfrak{g}$ is of Heisenberg type and W is a complementary subspace to $\mathbf{C}$ in $\mathfrak{g}$, we may define a bilinear map, $\omega : W \times W \to \mathbf{C}$, by $\omega(w, w') = [w, w']$ for all $w, w' \in W$. Then for $\xi_i := w_i + c_i \in W \oplus \mathbf{C} = \mathfrak{g}$, $i = 1, 2$, we have

$$[\xi_1, \xi_2] = [w_1 + c_1, w_2 + c_2] = 0 + \omega(w_1, w_2).$$

If we now suppose G is a finite-dimensional Lie group with Lie algebra $\mathfrak{g}$, then by the Baker–Campbell–Dynkin–Hausdorff formula

$$e^{\xi_1} e^{\xi_2} = e^{\xi_1 + \xi_2 + \frac{1}{2}[\xi_1, \xi_2]} = e^{w_1 + w_2 + c_1 + c_2 + \frac{1}{2}\omega(w_1, w_2)}.$$

In particular, we may introduce a group structure on $\mathfrak{g}$ by defining

$$(w_1 + c_1) \cdot (w_2 + c_2) = w_1 + w_2 + c_1 + c_2 + \frac{1}{2}\omega(w_1, w_2).$$

With this as motivation, we are now going to introduce a class of Heisenberg type Lie groups based on the following data.

Notation 3.1. Let (W, H, μ) be an abstract Wiener space, $\mathbf{C}$ be a finite-dimensional inner product space, and $\omega : W \times W \to \mathbf{C}$ be a continuous skew symmetric bilinear quadratic form on W. Further let

$$\|\omega\|_0 := \sup\bigl\{\bigl\|\omega(w_1, w_2)\bigr\|_{\mathbf{C}}:\ w_1, w_2 \in W \text{ with } \|w_1\|_W = \|w_2\|_W = 1\bigr\}. \tag{3.1}$$

be the uniform norm on ω which is finite by the assumed continuity of ω.

We now define $\mathfrak{g} := W \times \mathbf{C}$ which is a Banach space in the norm

$$\bigl\|(w, c)\bigr\|_{\mathfrak{g}} := \|w\|_W + \|c\|_{\mathbf{C}}. \tag{3.2}$$

We further define $\mathfrak{g}_{CM} := H \times \mathbf{C}$ which is a Hilbert space relative to the inner product

$$\bigl\langle (A, a), (B, b)\bigr\rangle_{\mathfrak{g}_{CM}} := \langle A, B\rangle_H + \langle a, b\rangle_{\mathbf{C}}. \tag{3.3}$$

The associated Hilbertian norm on $\mathfrak{g}_{CM}$ is given by

$$\bigl\|(A, a)\bigr\|_{\mathfrak{g}_{CM}} := \sqrt{\|A\|_H^2 + \|a\|_{\mathbf{C}}^2}. \tag{3.4}$$

It is easily checked that defining

$$\big[(w_1,c_1),(w_2,c_2)\big] := \big(0,\omega(w_1,w_2)\big) \tag{3.5}$$

for all $(w_1,c_1),(w_2,c_2)\in\mathfrak{g}$ makes $\mathfrak{g}$ into a Lie algebra such that $\mathfrak{g}_{CM}$ is Lie subalgebra of $\mathfrak{g}$. Note that this definition implies that $\mathbf{C}=[\mathfrak{g},\mathfrak{g}]$ is contained in the center of $\mathfrak{g}$. It is also easy to verify that we may make $\mathfrak{g}$ into a group using the multiplication rule

$$(w_1,c_1)\cdot(w_2,c_2)=\left(w_1+w_2,c_1+c_2+\frac{1}{2}\omega(w_1,w_2)\right). \tag{3.6}$$

The latter equations may be more simply expressed as

$$g_1g_2=g_1+g_2+\frac{1}{2}[g_1,g_2], \tag{3.7}$$

where $g_i=(w_i,c_i)$, $i=1,2$. As sets G and $\mathfrak{g}$ are the same.

The identity in G is $\mathbf{e}=(0,0)$ and the inverse is given by $g^{-1}=-g$ for all $g=(w,c)\in G$. Let us observe that $\{0\}\times\mathbf{C}$ is in the center of both G and $\mathfrak{g}$ and for h in the center of G, $g\cdot h=g+h$. In particular, since $[g,h]\in\{0\}\times\mathbf{C}$ it follows that $k\cdot[g,h]=k+[g,h]$ for all $k,g,h\in G$.

Definition 3.2. When we want to emphasize the group structure on $\mathfrak{g}$ we denote $\mathfrak{g}$ by G or $G(\omega)$. Similarly, when we view $\mathfrak{g}_{CM}$ as a subgroup of G it will be denoted by G_{CM} and will be called the *Cameron–Martin* subgroup.

Lemma 3.3. *The Banach space topologies on $\mathfrak{g}$ and $\mathfrak{g}_{CM}$ make G and G_{CM} into topological groups.*

Proof. Since $g^{-1}=-g$, the map $g\mapsto g^{-1}$ is continuous in the $\mathfrak{g}$ and $\mathfrak{g}_{CM}$ topologies. Since $(g_1,g_2)\mapsto g_1+g_2$ and $(g_1,g_2)\mapsto[g_1,g_2]$ are continuous in both the $\mathfrak{g}$ and $\mathfrak{g}_{CM}$ topologies, it follows from Eq. (3.7) that $(g_1,g_2)\mapsto g_1\cdot g_2$ is continuous as well. $\square$

For later purposes it is useful to observe, by Eqs. (3.5) and (3.7), that

$$\|g_1g_2\|_{\mathfrak{g}}\leqslant\|g_1\|_{\mathfrak{g}}+\|g_2\|_{\mathfrak{g}}+\frac{1}{2}\|\omega\|_0\|g_1\|_{\mathfrak{g}}\|g_2\|_{\mathfrak{g}}\quad\text{for any } g_1,g_2\in G. \tag{3.8}$$

Notation 3.4. To each $g\in G$, let $l_g:G\to G$ and $r_g:G\to G$ denote left and right multiplication by g, respectively.

Notation 3.5 *(Linear differentials).* Suppose $f:G\to\mathbb{C}$ is a Fréchet smooth function. For $g\in G$ and $h,k\in\mathfrak{g}$ let

$$f'(g)h:=\partial_h f(g)=\frac{d}{dt}\bigg|_0 f(g+th)$$

and

$$f''(g)(h \otimes k) := \partial_h \partial_k f(g).$$

Here and in the sequel a prime on a symbol will be used denote its derivative or differential.

As G is a vector space, to each $g \in G$ we can associate the tangent space (as in the following notation) to G at g, $T_g G$, which is naturally isomorphic to G.

Notation 3.6. For $v, g \in G$, let $v_g \in T_g G$ denote the tangent vector satisfying, $v_g f = f'(g)v$ for all Fréchet smooth functions, $f : G \to \mathbb{C}$.

We will write $\mathfrak{g}$ and $\mathfrak{g}_{\mathrm{CM}}$ for $T_{\mathbf{e}}G$ and $T_{\mathbf{e}}G_{\mathrm{CM}}$, respectively. Of course as sets we may view $\mathfrak{g}$ and $\mathfrak{g}_{\mathrm{CM}}$ as G and G_{CM}, respectively. For $h \in \mathfrak{g}$, let $\tilde{h}$ be the *left-invariant vector field* on G such that $\tilde{h}(g) = h$ when $g = \mathbf{e}$. More precisely if $\sigma(t) \in G$ is any smooth curve such that $\sigma(0) = \mathbf{e}$ and $\dot{\sigma}(0) = h$ (e.g. $\sigma(t) = th$), then

$$\tilde{h}(g) = l_{g*}h := \frac{d}{dt}\bigg|_0 g \cdot \sigma(t). \tag{3.9}$$

As usual we view $\tilde{h}$ as a first-order differential operator acting on smooth functions, $f : G \to \mathbb{C}$, by

$$(\tilde{h} f)(g) = \frac{d}{dt}\bigg|_0 f\big(g \cdot \sigma(t)\big). \tag{3.10}$$

Proposition 3.7. *Let $f : G \to \mathbb{C}$ be a smooth function, $h = (A, a) \in \mathfrak{g}$ and $g = (w, c) \in G$. Then*

$$\tilde{h}(g) := l_{g*}h = \left(A, a + \frac{1}{2}\omega(w, A)\right)_g \quad \text{for all } g = (w, c) \in G \tag{3.11}$$

and in particular using Notation 3.6

$$\widetilde{(A, a)} f(g) = f'(g)\left(A, a + \frac{1}{2}\omega(w, A)\right). \tag{3.12}$$

Furthermore, if $h = (A, a)$, $k = (B, b)$, and then

$$(\tilde{h}\tilde{k} f - \tilde{k}\tilde{h} f) = \widetilde{[h, k]} f. \tag{3.13}$$

In other words, the Lie algebra structure on $\mathfrak{g}$ induced by the Lie algebra structure on the left-invariant vector fields on G is the same as the Lie algebra structure defined in Eq. (3.5).

Proof. Since $th = t(A,a)$ is a curve in G passing through the identity at $t=0$, we have

$$\begin{aligned}\tilde{h}(g) &= \frac{d}{dt}\bigg|_0 \big[g\cdot(th)\big] = \frac{d}{dt}\bigg|_0 \big[(w,c)\cdot t(A,a)\big] \\ &= \frac{d}{dt}\bigg|_0 \left[\left(w+tA, c+ta+\frac{t}{2}\omega(w,A)\right)\right] \\ &= \left(A, a+\frac{1}{2}\omega(w,A)\right).\end{aligned}$$

So by the chain rule, $(\tilde{h}f)(g) = f'(g)\tilde{h}(g)$ and hence

$$\begin{aligned}(\tilde{h}\tilde{k}f)(g) &= \frac{d}{dt}\bigg|_0 \big[f'(g\cdot th)\tilde{k}(g\cdot th)\big] \\ &= f''(g)\big(\tilde{h}(g)\otimes\tilde{k}(g)\big) + f'(g)\frac{d}{dt}\bigg|_0 \tilde{k}(g\cdot th),\end{aligned} \tag{3.14}$$

where

$$\frac{d}{dt}\bigg|_0 \tilde{k}(g\cdot th) = \frac{d}{dt}\bigg|_0 \left(B, a+\frac{1}{2}\omega(w+tA,B)\right) = \left(0, \frac{1}{2}\omega(A,B)\right).$$

Since $f''(g)$ is symmetric, it now follows by subtracting Eq. (3.14) with h and k interchanged from itself that

$$(\tilde{h}\tilde{k}f - \tilde{k}\tilde{h}f)(g) = f'(g)\big(0,\omega(A,B)\big) = f'(g)[h,k] = \big(\widetilde{[h,k]}f\big)(g)$$

as desired. □

Lemma 3.8. *The one parameter group in G, e^{th}, determined by $h=(A,a)\in\mathfrak{g}$, is given by*

$$e^{th} = th = t(A,a). \tag{3.15}$$

Proof. Letting $(w(t),c(t)) := e^{th}$, according to Eq. (3.11) we have that

$$\frac{d}{dt}\big(w(t),c(t)\big) = \left(A, a+\frac{1}{2}\omega\big(w(t),A\big)\right) \quad \text{with } w(0)=0 \text{ and } c(0)=0.$$

The solution to this differential equation is easily seen to be given by Eq. (3.15). □

3.1. Length and distance estimates

Notation 3.9. Let $T>0$ and C^1_{CM} denote the collection of C^1-paths, $g:[0,T]\to G_{\mathrm{CM}}$. The length of g is defined as

$$\ell_{G_{\mathrm{CM}}}(g) = \int_0^T \big\|l_{g^{-1}(s)*}g'(s)\big\|_{\mathfrak{g}_{\mathrm{CM}}}\,ds. \tag{3.16}$$

As usual, the Riemannian distance between $x, y \in G_{\mathrm{CM}}$ is defined as

$$d_{G_{\mathrm{CM}}}(x, y) = \inf\{\ell_{G_{\mathrm{CM}}}(g)\colon\ g \in C^1_{\mathrm{CM}} \text{ such that } g(0) = x \text{ and } g(T) = y\}.$$

It will also be convenient to define $|y| := d_{G_{\mathrm{CM}}}(\mathbf{e}, y)$ for all $y \in G_{\mathrm{CM}}$. (The value of $T > 0$ used in defining $d_{C_{\mathrm{CM}}}$ is irrelevant since the length functional is reparametrization invariant.)

Let

$$C := \sup\{\|\omega(h,k)\|_{\mathbf{C}}\colon\ \|h\|_H = \|k\|_H = 1\} \leqslant C_2\|\omega\|_0 < \infty. \tag{3.17}$$

The inequality in Eq. (3.17) is a consequence of Eq. (2.10) and the definition of $\|\omega\|_0$ in Eq. (3.1).

Proposition 3.10. *Let $\varepsilon := 1/C$ where C is as in Eq.* (3.17). *Then for all $x, y \in G_{\mathrm{CM}}$,*

$$d_{G_{\mathrm{CM}}}(x, y) \leqslant \left(1 + \frac{C}{2}\|x\|_{\mathfrak{g}_{\mathrm{CM}}} \wedge \|y\|_{\mathfrak{g}_{\mathrm{CM}}}\right)\|y - x\|_{\mathfrak{g}_{\mathrm{CM}}} \tag{3.18}$$

and in particular, $|x| = d_{G_{\mathrm{CM}}}(\mathbf{e}, x) \leqslant \|x\|_{\mathfrak{g}_{\mathrm{CM}}}$. Moreover, there exists $K < \infty$ such that if $x, y \in G_{\mathrm{CM}}$ with $d_{G_{\mathrm{CM}}}(x, y) < \varepsilon/2 = 1/2C$, then

$$\|y - x\|_{\mathfrak{g}_{\mathrm{CM}}} \leqslant K(1 + \|x\|_{\mathfrak{g}_{\mathrm{CM}}} \wedge \|y\|_{\mathfrak{g}_{\mathrm{CM}}})d_{G_{\mathrm{CM}}}(x, y). \tag{3.19}$$

As a consequence of Eqs. (3.18) *and* (3.19) *we see that the topology on G_{CM} induced by $d_{G_{\mathrm{CM}}}$ is the same as the Hilbert topology induced by $\|\cdot\|_{\mathfrak{g}_{\mathrm{CM}}}$.*

Remark 3.11. The equivalence of these two topologies in an infinite-dimensional setting has been addressed in [24] in the case of Hilbert–Schmidt groups of operators.

Proof. For notational simplicity, let $T = 1$. If $g(s) = (w(s), a(s))$ is a path in C^1_{CM} for $0 \leqslant s \leqslant 1$, then by Eq. (3.11)

$$\begin{aligned} l_{g^{-1}(s)*}g'(s) &= \left(w'(s), a'(s) - \frac{1}{2}\omega(w(s), w'(s))\right) \\ &= g'(s) - \frac{1}{2}[g(s), g'(s)] \end{aligned} \tag{3.20}$$

and we may write Eq. (3.16) more explicitly as

$$\ell_{G_{\mathrm{CM}}}(g) = \int_0^1 \left\|g'(s) - \frac{1}{2}[g(s), g'(s)]\right\|_{\mathfrak{g}_{\mathrm{CM}}} ds. \tag{3.21}$$

If we now apply Eq. (3.21) to $g(s) = x + s(y - x)$ for $0 \leqslant s \leqslant 1$, we see that

$$d_{G_{CM}}(x,y) \leqslant \ell_{G_{CM}}(g) = \int_0^1 \left\| (y-x) - \frac{1}{2}\big[x + s(y-x), (y-x)\big] \right\|_{\mathfrak{g}_{CM}} ds$$
$$= \left\| (y-x) - \frac{1}{2}\big[x, (y-x)\big] \right\|_{\mathfrak{g}_{CM}} \leqslant \left(1 + \frac{C}{2}\|x\|_{\mathfrak{g}_{CM}}\right)\|y-x\|_{\mathfrak{g}_{CM}}.$$

As we may interchange the roles of x and y in this inequality, the proof of Eq. (3.18) is complete.

Let

$$B_\varepsilon := \big\{x \in \mathfrak{g}_{CM}\colon \|x\|_{\mathfrak{g}_{CM}} \leqslant \varepsilon\big\},$$

$y \in B_\varepsilon$, and $g : [0,1] \to G_{CM}$ be a C^1-path such that $g(0) = (0,0) = \mathbf{e}$ and $g(1) = y$. Further let $T \in [0,1]$ be the first time that g exits B_ε with the convention that $T = 1$ if $g([0,1]) \subset B_\varepsilon$. Then from Eq. (3.21)

$$\begin{aligned}
\ell_{G_{CM}}(g) &\geqslant \ell_{G_{CM}}(g|_{[0,T]}) \\
&\geqslant \int_0^T \left[\|g'(s)\|_{\mathfrak{g}_{CM}} - \frac{1}{2}\big\|\big[g(s), g'(s)\big]\big\|_{\mathfrak{g}_{CM}} \right] ds \\
&\geqslant \left(1 - \frac{C}{2}\varepsilon\right) \cdot \int_0^T \|g'(s)\|_{\mathfrak{g}_{CM}}\, ds \geqslant \left(1 - \frac{C}{2}\varepsilon\right) \cdot \|g(T)\|_{\mathfrak{g}_{CM}} \\
&\geqslant \frac{1}{2}\|g(T)\|_{\mathfrak{g}_{CM}} \geqslant \frac{1}{2}\|y\|_{\mathfrak{g}_{CM}}.
\end{aligned} \tag{3.22}$$

Optimizing Eq. (3.22) over g implies

$$|y| = d_{G_{CM}}(\mathbf{e}, y) \geqslant \frac{1}{2}\|y\|_{\mathfrak{g}_{CM}} \quad \text{for all } y \in B_\varepsilon.$$

If in the above argument y was not in B_ε, then the path g would have had to exit B_ε and we could conclude that $\ell_{G_{CM}}(g) \geqslant \|g(T)\|_{\mathfrak{g}_{CM}}/2 = \varepsilon/2$ and therefore that $d_{G_{CM}}(\mathbf{e}, y) \geqslant \varepsilon/2$. Hence we have shown that

$$|y| = d_{G_{CM}}(\mathbf{e}, y) \geqslant \frac{1}{2}\min\big(\varepsilon, \|y\|_{\mathfrak{g}_{CM}}\big) \quad \text{for all } y \in G_{CM}.$$

Now suppose that $x, y \in G_{CM}$ and (without loss of generality) that $\|x\|_{\mathfrak{g}_{CM}} \leqslant \|y\|_{\mathfrak{g}_{CM}}$. Using the left-invariance of $d_{G_{CM}}$, it follows that

$$d_{G_{CM}}(x,y) = d_{G_{CM}}\big(\mathbf{e}, x^{-1}y\big) \geqslant \frac{1}{2}\min\big(\varepsilon, \|x^{-1}y\|_{\mathfrak{g}_{CM}}\big). \tag{3.23}$$

If we further suppose that $d_{G_{CM}}(x,y) < \frac{\varepsilon}{2}$, we may conclude from Eq. (3.23) that

$$\left\| y - x - \frac{1}{2}[x,y] \right\|_{\mathfrak{g}_{CM}} = \|x^{-1}y\|_{\mathfrak{g}_{CM}} \leqslant 2d_{G_{CM}}(x,y).$$

If we write $x = (A, a)$ and $y = (B, b)$, it follows that

$$\|B - A\|_H^2 + \left\|b - a - \frac{1}{2}\omega(A, B)\right\|_{\mathbf{C}}^2 \leqslant 4d_{G_{\mathrm{CM}}}^2(x, y)$$

and therefore $\|B - A\|_H \leqslant 2d_{G_{\mathrm{CM}}}(x, y)$ and

$$\begin{aligned}\|b - a\|_{\mathbf{C}} &\leqslant \left\|b - a - \frac{1}{2}\omega(A, B)\right\|_{\mathbf{C}} + \left\|\frac{1}{2}\omega(A, B)\right\|_{\mathbf{C}} \\ &\leqslant 2d_{G_{\mathrm{CM}}}(x, y) + \frac{1}{2}\left\|\omega(A, B - A)\right\|_{\mathbf{C}} \\ &\leqslant 2d_{G_{\mathrm{CM}}}(x, y) + \frac{C}{2}\|A\|_H\|B - A\|_H \\ &\leqslant 2d_{G_{\mathrm{CM}}}(x, y)\left(1 + \frac{C}{2}\|A\|_H\right) \leqslant 2d_{G_{\mathrm{CM}}}(x, y)\left(1 + \frac{C}{2}\|x\|_{\mathfrak{g}_{\mathrm{CM}}}\right).\end{aligned}$$

Combining these results shows that if $d_{G_{\mathrm{CM}}}(x, y) < \frac{\varepsilon}{2}$ then

$$\|y - x\|_{\mathfrak{g}_{\mathrm{CM}}}^2 \leqslant 4d_{G_{\mathrm{CM}}}^2(x, y)\left(1 + \left(1 + \frac{C}{2}\|x\|_{\mathfrak{g}_{\mathrm{CM}}}\right)^2\right)$$

from which Eq. (3.19) easily follows. □

We are most interested in the case where $\{\omega(A, B)\colon\ A, B \in H\}$ is a total subset of $\mathbf{C}$, i.e. $\operatorname{span}\{\omega(A, B)\colon\ A, B \in H\} = \mathbf{C}$. In this case it turns out that straight line paths are bad approximations to the geodesics joining $\mathbf{e} \in G_{\mathrm{CM}}$ to points $x \in G_{\mathrm{CM}}$ far away from $\mathbf{e}$. For points $x \in G_{\mathrm{CM}}$ distant from $\mathbf{e}$ it is better to use "horizontal" paths instead which leads to the following distance estimates.

Theorem 3.12. *Suppose that $\{\omega(A, B)\colon\ A, B \in H\}$ is a total subset of $\mathbf{C}$. Then there exists $C(\omega) < \infty$ such that*

$$d_{\mathrm{CM}}\big(\mathbf{e}, (A, a)\big) \leqslant C(\omega)\big(\|A\|_H + \sqrt{\|a\|_{\mathbf{C}}}\big) \quad \text{for all } (A, a) \in \mathfrak{g}_{\mathrm{CM}}. \tag{3.24}$$

Moreover, for any $\varepsilon_0 > 0$ there exists $\gamma(\varepsilon_0) > 0$ such that and

$$\gamma(\varepsilon_0)\big(\|A\|_H + \sqrt{\|a\|_{\mathbf{C}}}\big) \leqslant d_{\mathrm{CM}}\big(\mathbf{e}, (A, a)\big) \quad \text{if } d_{\mathrm{CM}}\big(\mathbf{e}, (A, a)\big) \geqslant \varepsilon_0. \tag{3.25}$$

Thus away from any neighborhood of the identity, $d_{\mathrm{CM}}(\mathbf{e}, (A, a))$ is comparable to $\|A\|_H + \sqrt{\|a\|_{\mathbf{C}}}$.

Since this theorem is not central to the rest of the paper we will relegate its proof to Appendix C. The main point of Theorem 3.12 is to explain why Theorem 4.16 is an infinite-dimensional analogue of the integrated Gaussian heat kernel bound in Eq. (1.5).

3.2. Norm estimates

Notation 3.13. Suppose H and $\mathbf{C}$ are real (complex) Hilbert spaces, $L: H \to \mathbf{C}$ is a bounded operator, $\omega: H \times H \to \mathbf{C}$ is a continuous (complex) bilinear form, and $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis for H. The Hilbert–Schmidt norms of L and ω are defined by

$$\|L\|_{H^* \otimes \mathbf{C}}^2 := \sum_{j=1}^{\infty} \|Le_j\|_{\mathbf{C}}^2, \tag{3.26}$$

and

$$\|\omega\|_2^2 = \|\omega\|_{H^* \otimes H^* \otimes \mathbf{C}} := \sum_{i,j=1}^{\infty} \|\omega(e_i, e_j)\|_{\mathbf{C}}^2. \tag{3.27}$$

It is easy to verify directly that these definitions are basis independent. Also see Eq. (3.29) below.

Proposition 3.14. *Suppose that (W, H, μ) is a real abstract Wiener space, $\omega: W \times W \to \mathbf{C}$ is as in Notation* 3.1*, and $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis for H. Then*

$$\|\omega(w, \cdot)\|_{H^* \otimes \mathbf{C}}^2 \leqslant C_2 \|\omega\|_0^2 \|w\|_W^2 \quad \text{for all } w \in W \tag{3.28}$$

and

$$\|\omega\|_2^2 = \int_{W \times W} \|\omega(w, w')\|_{\mathbf{C}}^2 \, d\mu(w) \, d\mu(w') \leqslant \|\omega\|_0^2 C_2^2 < \infty, \tag{3.29}$$

where C_2 is as in Eq. (2.2)*.*

Proof. From Eq. (2.13),

$$\begin{aligned} \|\omega(w, \cdot)\|_{H^* \otimes \mathbf{C}}^2 &= \int_W \|\omega(w, w')\|_{\mathbf{C}}^2 \, d\mu(w') \\ &\leqslant \|\omega\|_0^2 \|w\|_W^2 \int_W \|w'\|_W^2 \, d\mu(w') = C_2 \|\omega\|_0^2 \|w\|_W^2. \end{aligned}$$

Similarly, viewing $w \to \omega(w, \cdot)$ as a continuous linear map from W to $H^* \otimes \mathbf{C}$ it follows from Eqs. (2.13) and (2.14), that

$$\begin{aligned} \|\omega\|_2^2 &= \|h \mapsto \omega(h, \cdot)\|_{H^* \otimes (H^* \otimes \mathbf{C})}^2 = \int_W \|\omega(w, \cdot)\|_{H^* \otimes \mathbf{C}}^2 \, d\mu(w) \\ &\leqslant \int_W C_2 \|\omega\|_0^2 \|w\|_W^2 \, d\mu(w) = C_2^2 \|\omega\|_0^2. \qquad \square \end{aligned}$$

Remark 3.15. The Lie bracket on $\mathfrak{g}_{CM}$ has the following continuity property,

$$\big\|\big[(A,a),(B,b)\big]\big\|_{\mathfrak{g}_{CM}} \leqslant C\big\|(A,a)\big\|_{\mathfrak{g}_{CM}}\big\|(B,b)\big\|_{\mathfrak{g}_{CM}},$$

where $C \leqslant \|\omega\|_2$ as in Eq. (3.17). This is a consequence of the following simple estimates

$$\begin{aligned}\big\|\big[(A,a),(B,b)\big]\big\|_{\mathfrak{g}_{CM}} &= \big\|\big(0,\omega(A,B)\big)\big\|_{\mathfrak{g}_{CM}} = \big\|\omega(A,B)\big\|_{\mathbf{C}} \\ &\leqslant C\|A\|_H\|B\|_H \leqslant C\big\|(A,a)\big\|_{\mathfrak{g}_{CM}}\big\|(B,b)\big\|_{\mathfrak{g}_{CM}}.\end{aligned}$$

This continuity property of the Lie bracket is often used to prove that the exponential map is a local diffeomorphism (e.g. see [24] in the case of infinite-dimensional matrix groups). In the Heisenberg group setting the exponential map is a global diffeomorphism as follows from Lemma 3.8, where we have not used continuity of the Lie bracket.

Lemma 3.16. *Suppose that H is a real Hilbert space, $\mathbf{C}$ is a real finite-dimensional inner product space, and $\ell : H \to \mathbf{C}$ is a continuous linear map. Then for any orthonormal basis $\{e_j\}_{j=1}^{\infty}$ of H the series*

$$\sum_{j=1}^{\infty} \ell(e_j) \otimes \ell(e_j) \in \mathbf{C} \otimes \mathbf{C} \tag{3.30}$$

and

$$\sum_{j=1}^{\infty} \ell(e_j) \otimes e_j \in \mathbf{C} \otimes H \tag{3.31}$$

are convergent and independent of the basis.[3]

Proof. If $\{f_i\}_{i=1}^{\dim \mathbf{C}}$ is an orthonormal basis for $\mathbf{C}$, then

$$\begin{aligned}\sum_{j=1}^{\infty}\big\|\ell(e_j)\otimes\ell(e_j)\big\|_{\mathbf{C}\otimes\mathbf{C}} &= \sum_{j=1}^{\infty}\big\|\ell(e_j)\big\|_{\mathbf{C}}^2 \\ &= \sum_{i=1}^{\dim \mathbf{C}}\sum_{j=1}^{\infty}\big(f_i,\ell(e_j)\big)^2 = \sum_{i=1}^{\dim \mathbf{C}}\big\|\big(f_i,\ell(\cdot)\big)\big\|_{H^*}^2 < \infty\end{aligned}$$

which shows that the sum in Eq. (3.30) is absolutely convergent and that ℓ is Hilbert–Schmidt. Similarly, since $\{\ell(e_j)\otimes e_j\}_{j=1}^{\infty}$ is an orthogonal set in $\mathbf{C}\otimes H$ and

$$\sum_{j=1}^{\infty}\big\|\ell(e_j)\otimes e_j\big\|_{\mathbf{C}\otimes H}^2 = \sum_{j=1}^{\infty}\big\|\ell(e_j)\big\|_{\mathbf{C}}^2 < \infty,$$

the sum in Eq. (3.31) is convergent as well.

[3] If we were to allow $\mathbf{C}$ to be an infinite-dimensional Hilbert space here, we would have to assume that ℓ is Hilbert–Schmidt. When $\dim \mathbf{C} < \infty$, $\ell : H \to \mathbf{C}$ is Hilbert–Schmidt iff it is bounded.

Recall that if H and K are two real Hilbert spaces then the Hilbert space tensor product, $H \otimes K$, is unitarily equivalent to the space of Hilbert–Schmidt operators, $HS(H, K)$, from H to K. Under this identification, $h \otimes k \in H \otimes K$ corresponds to the operator (still denoted by $h \otimes k$) in $HS(H, K)$ defined by; $H \ni h' \to (h, h')_H k \in K$. Using this identification we have that for all $c \in \mathbf{C}$;

$$\left(\sum_{j=1}^{\infty} \ell(e_j) \otimes \ell(e_j)\right) c = \sum_{j=1}^{\infty} \ell(e_j) \langle \ell(e_j), c \rangle_{\mathbf{C}} = \sum_{j=1}^{\infty} \ell(e_j) \langle e_j, \ell^* c \rangle_{\mathbf{C}} = \ell\left(\sum_{j=1}^{\infty} \langle e_j, \ell^* c \rangle_{\mathbf{C}} e_j\right) = \ell \ell^* c$$

and

$$\left(\sum_{j=1}^{\infty} \ell(e_j) \otimes e_j\right) c = \sum_{j=1}^{\infty} e_j \langle \ell(e_j), c \rangle_{\mathbf{C}} = \sum_{j=1}^{\infty} e_j \langle e_j, \ell^* c \rangle_{\mathbf{C}} = \ell^* c,$$

which clearly shows that Eqs. (3.30) and(3.31) are basis-independent. □

3.3. Examples

Here we describe several examples including finite-dimensional Heisenberg groups. As we mentioned earlier a typical example of such a group is the Heisenberg group of a symplectic vector space. For each of the examples presented we will explicitly compute the norm $\|\omega\|_2^2$ of the form ω as defined in Eq. (3.27). In Section 7 we will also explicitly compute the Ricci tensor for each of the examples introduced here.

To describe some of the examples below, it is convenient to use complex Banach and Hilbert spaces. However, for the purposes of this paper the complex structure on these spaces should be forgotten. In doing so we will use the following notation. If X is a complex vector space, let X_{Re} denote X thought of as a real vector space. If $(H, \langle \cdot,\cdot \rangle_H)$ is a complex Hilbert space, we define $\langle \cdot,\cdot \rangle_{H_{\mathrm{Re}}} = \mathrm{Re}\langle \cdot,\cdot \rangle_H$ in which case $(H_{\mathrm{Re}}, \langle \cdot,\cdot \rangle_{H_{\mathrm{Re}}})$ becomes a real Hilbert space. Before going to the examples, let us record the relationship between the complex and real Hilbert–Schmidt norms of Notation 3.13.

Lemma 3.17. *Suppose H and $\mathbf{C}$ are complex Hilbert spaces, $L : H \to \mathbf{C}$ and $\omega : H \times H \to \mathbf{C}$ are as in Notation* 3.13*, and $c \in \mathbf{C}$. Then*

$$\|L\|^2_{H^*_{\mathrm{Re}} \otimes \mathbf{C}_{\mathrm{Re}}} = 2\|L\|^2_{H^* \otimes \mathbf{C}}, \tag{3.32}$$

$$\left\|\langle \omega(\cdot,\cdot), c \rangle_{\mathbf{C}_{\mathrm{Re}}}\right\|^2_{H^*_{\mathrm{Re}} \otimes H^*_{\mathrm{Re}}} = 2\left\|\langle \omega(\cdot,\cdot), c \rangle_{\mathbf{C}}\right\|^2_{H^* \otimes H^*}, \tag{3.33}$$

and

$$\left\|\omega(\cdot,\cdot)\right\|^2_{H^*_{\mathrm{Re}} \otimes H^*_{\mathrm{Re}} \otimes \mathbf{C}_{\mathrm{Re}}} = 4\left\|\omega(\cdot,\cdot)\right\|^2_{H^* \otimes H^* \otimes \mathbf{C}}. \tag{3.34}$$

Proof. A straightforward proof. □

Example 3.18 *(Finite-dimensional real Heisenberg group).* Let $\mathbf{C} = \mathbb{R}$, $W = H = (\mathbb{C}^n)_{\mathrm{Re}} \cong \mathbb{R}^{2n}$, and $\omega(w,z) := \operatorname{Im}\langle w, z\rangle$ be the standard symplectic form on $\mathbb{R}^{2n}$, where $\langle w, z\rangle = w \cdot \bar{z}$ is the usual inner product on $\mathbb{C}^n$. Any element of the group $\mathbf{H}^n_{\mathbb{R}} := G(\omega)$ can be written as $g = (z, c)$, where $z \in \mathbb{C}^n$ and $c \in \mathbb{R}$. As above, the Lie algebra, $\mathfrak{h}^n_{\mathbb{R}}$, of $\mathbf{H}^n_{\mathbb{R}}$ is, as a set, equal to $\mathbf{H}^n_{\mathbb{R}}$ itself. If $\{e_j\}_{j=1}^n$ is an orthonormal basis for $\mathbb{R}^n$ then $\{e_j, ie_j\}_{j=1}^n$ is an orthonormal basis for H and (real) Hilbert–Schmidt norm of ω is given by

$$\|\omega\|^2_{H^*\otimes H^*} = \sum_{j,k=1}^n \sum_{\varepsilon,\delta\in\{1,i\}} \left[\operatorname{Im}\langle \varepsilon e_j, \delta e_k\rangle\right]^2 = \sum_{j,k=1}^n 2\delta_{j,k} = 2n. \tag{3.35}$$

Example 3.19 *(Finite-dimensional complex Heisenberg group).* Suppose that $W = H = \mathbb{C}^n \times \mathbb{C}^n$, $\mathbf{C} = \mathbb{C}$, and $\omega\colon W \times W \to \mathbb{C}$ is defined by

$$\omega\big((w_1, w_2), (z_1, z_2)\big) = w_1 \cdot z_2 - w_2 \cdot z_1.$$

Any element of the group $\mathbf{H}^n_{\mathbb{C}} := G(\omega)$ can be written as $g = (z_1, z_2, c)$, where $z_1, z_2 \in \mathbb{C}^n$ and $c \in \mathbb{C}$. As above, the Lie algebra, $\mathfrak{h}^n_{\mathbb{C}}$, of $\mathbf{H}^n_{\mathbb{C}}$ is, as a set, equal to $\mathbf{H}^n_{\mathbb{C}}$ itself. In this case $\{(e_j, 0), (0, e_j)\}_{j=1}^n$ is a complex orthonormal basis for H. The (complex) Hilbert–Schmidt norm of the symplectic form ω is given by

$$\|\omega\|^2_{H^*\otimes H^*} = 2\sum_{j=1}^n \big|\omega\big((e_j, 0), (0, e_j)\big)\big|^2 = 2n.$$

Example 3.20. Let $(K, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and Q be a strictly positive trace class operator on K. For $h, k \in K$, let $\langle h, k\rangle_Q := \langle h, Qk\rangle$ and $\|h\|_Q := \sqrt{\langle h, h\rangle_Q}$. Also let $(K_Q, \langle\cdot,\cdot\rangle_Q)$ denote the Hilbert space completion of $(K, \|\cdot\|_Q)$. Analogous to Example 3.18, let $H := K_{\mathrm{Re}}$, $W := (K_Q)_{\mathrm{Re}}$, and $\omega : W \times W \to \mathbb{R} =: \mathbf{C}$ be defined by

$$\omega(w, z) = \operatorname{Im}\langle w, z\rangle_Q \quad \text{for all } w, z \in W.$$

Then $G(\omega) = W \times \mathbb{R}$ is a real group and (W, H) determines an abstract Wiener space (see for example [36, Exercise 17, p. 59] and [8, Example 3.9.7]). Let $\{e_j\}_{j=1}^\infty$ be an orthonormal basis for K so that $\{e_j, ie_j\}_{j=1}^\infty$ is an orthonormal basis for $(H, \operatorname{Re}\langle\cdot,\cdot\rangle_K)$. Then the real Hilbert–Schmidt norm of ω is given by

$$\begin{aligned}
\|\omega\|^2_{H^*\otimes H^*} &= \sum_{j,k=1}^\infty \sum_{\varepsilon,\delta\in\{1,i\}} \left[\operatorname{Im}^2\langle \varepsilon e_j, Q\delta e_k\rangle\right] \\
&= 2\sum_{j,k=1}^\infty \left[\operatorname{Im}^2\langle e_j, Qe_k\rangle + \operatorname{Re}^2\langle e_j, Qe_k\rangle\right] = 2\sum_{j,k=1}^\infty \left|\langle e_j, Qe_k\rangle\right|^2 \\
&= 2\sum_{k=1}^\infty \|Qe_k\|^2 = 2\|Q\|^2_{HS} = 2\operatorname{tr} Q^2.
\end{aligned} \tag{3.36}$$

Example 3.21. Again let $(K, \langle\cdot,\cdot\rangle)$, Q, and $(K_Q, \langle\cdot,\cdot\rangle_Q)$ be as in the previous example. Let us further assume that K is equipped with a conjugation, $k \to \bar{k}$, which is isometric and commutes with Q. Analogously to Example 3.19, let $W := K_Q \times K_Q$, $H = K \times K$, and let $\omega: W \times W \to \mathbb{C}$ be defined by

$$\omega\big((w_1, w_2), (z_1, z_2)\big) = \langle w_1, \bar{z}_2\rangle_Q - \langle w_2, \bar{z}_1\rangle_Q,$$

which is skew symmetric because the conjugation commutes with Q. If $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis for K, then $\{\bar{e}_j\}_{j=1}^{\infty}$ is also an orthonormal basis for K (because the conjugation is isometric) and $\{(e_j, 0), (0, e_j)\}_{j=1}^{\infty}$ is a orthonormal basis for H. Hence, the (complex) Hilbert–Schmidt norm of ω is given by

$$\begin{aligned}\|\omega\|_{H^*\otimes H^*}^2 &= \sum_{j,k=1}^{\infty}\Big(\big|\omega\big((e_j, 0), (0, e_k)\big)\big|^2 + \big|\omega\big((0, e_k), (e_j, 0)\big)\big|^2\Big)\\ &= 2\sum_{j,k=1}^{\infty}\big|\langle e_j, Q\bar{e}_k\rangle\big|^2 = 2\sum_{k=1}^{\infty}\|Q\bar{e}_k\|^2 = 2\|Q\|_{HS}^2 = 2\operatorname{tr} Q^2. \end{aligned} \tag{3.37}$$

Example 3.22. Suppose that $(V, \langle\cdot,\cdot\rangle_V)$ is a d-dimensional $\mathbb{F}$-inner product space ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$), $\mathbf{C}$ is a finite-dimensional $\mathbb{F}$-inner product space, $\alpha: V \times V \to \mathbf{C}$ is an anti-symmetric bilinear form on V, and $\{q_j\}_{j=1}^{\infty}$ is a sequence of positive numbers such that $\sum_{j=1}^{\infty} q_j < \infty$. Let

$$W = \left\{v \in V^{\mathbb{N}}\colon \sum_{j=1}^{\infty} q_j\|v_j\|_V^2 < \infty\right\} \quad \text{and}$$

$$H = \left\{v \in V^{\mathbb{N}}\colon \sum_{j=1}^{\infty}\|v_j\|_V^2 < \infty\right\} \subset W,$$

each of which are Hilbert spaces when equipped with the inner products

$$\langle v, w\rangle_W := \sum_{j=1}^{\infty} q_j\langle v_j, w_j\rangle_V \quad \text{and}$$

$$\langle v, w\rangle_H := \sum_{j=1}^{\infty}\langle v_j, w_j\rangle_V,$$

respectively. Further let $\omega: W \times W \to \mathbf{C}$ be defined by

$$\omega(v, w) = \sum_{j=1}^{\infty} q_j\alpha(v_j, w_j).$$

Then $(W_{\mathrm{Re}}, H_{\mathrm{Re}})$ is an abstract Wiener space (see for example [36, Exercise 17, p. 59] and [8, Example 3.9.7]) and, since

$$\begin{aligned}\left|\omega(v,w)\right| &= \sum_{j=1}^{\infty} q_j \left|\alpha(v_j,w_j)\right| \leqslant \|\alpha\|_0 \sum_{j=1}^{\infty} q_j \|v_j\|_V \|w_j\|_V \\ &\leqslant \|\alpha\|_0 \|v\|_W \|w\|_W,\end{aligned}$$

we have $\|\omega\|_0 \leqslant \|\alpha\|_0$. For $v \in V$, let $v(j) := (0,\dots,0,v,0,0,\dots) \in H$ where the v is put in the jth position. If $\{u_a\}_{a=1}^d$ is an orthonormal basis for V, then $\{u_a(j)\colon a=1,\dots,d\}_{j=1}^{\infty}$ is an orthonormal basis for H. Therefore,

$$\begin{aligned}\left\|\langle\omega(\cdot,\cdot),c\rangle_{\mathbf{C}}\right\|^2_{H^*\otimes H^*} &= \sum_{j,k=1}^{\infty}\sum_{a,b=1}^{d} \left\langle\omega\big(u_a(j),u_b(k)\big),c\right\rangle^2_{\mathbf{C}} \\ &= \sum_{j=1}^{\infty}\sum_{a,b=1}^{d} q_j^2 \langle\alpha(u_a,u_b),c\rangle^2_{\mathbf{C}} \\ &= \left(\sum_{j=1}^{\infty} q_j^2\right) \left\|\langle\alpha(\cdot,\cdot),c\rangle_{\mathbf{C}}\right\|^2_{V^*\otimes V^*} \quad \text{for all } c \in \mathbf{C}. \end{aligned} \tag{3.38}$$

Example 3.23. Let $(V, \langle\cdot,\cdot\rangle, \alpha)$ be as in Example 3.22 with $\mathbb{F} = \mathbb{C}$,

$$W = \big\{\sigma \in C\big([0,1],V\big)\colon \sigma(0) = 0\big\}$$

and H be the associated Cameron–Martin space,

$$H := H(V) = \left\{h \in W\colon \int_0^1 \left\|h'(s)\right\|_V^2 ds < \infty\right\},$$

wherein $\int_0^1 \|h'(s)\|_V^2\, ds := \infty$ if h is not absolutely continuous. Further let η be a complex measure on [0, 1] and

$$\omega(\sigma_1,\sigma_2) := \int_0^1 \alpha\big(\sigma_1(s),\sigma_2(s)\big)\, d\eta(s) \quad \text{for all } \sigma_1,\sigma_2 \in W.$$

Then (W, H, ω) satisfies all of the assumptions in Notation 3.1. Let $\{u_a\}_{a=1}^d$ be the orthonormal basis of V, $\{l_j\}_{j=1}^{\infty}$ be the orthonormal basis of $H(\mathbb{R})$, then $\{l_j u_a\colon a = 1,2,\dots,d\}_{j=1}^{\infty}$ is an orthonormal basis of H and (see [22, Lemma 3.8])

$$\sum_{j=1}^{\infty} l_j(s) l_j(t) = s \wedge t \quad \text{for all } s,t \in [0,1]. \tag{3.39}$$

If we let λ be the total variation of η, then $d\eta = \rho\, d\lambda$, where $\rho = \frac{d\eta}{d\lambda}$. Hence if $d\bar{\eta}(t) := \bar{\rho}(t)\, d\lambda(t)$ and $c \in \mathbf{C}$, then

$$\begin{aligned}
&\left\|\langle\omega(\cdot,\cdot),c\rangle_{\mathbf{C}}\right\|_2^2\\
&=\sum_{j,k=1}^{\infty}\sum_{a,b=1}^{d}\left|\langle\omega(l_ju_a,l_ku_b),c\rangle_{\mathbf{C}}\right|^2\\
&=\sum_{j,k=1}^{\infty}\sum_{a,b=1}^{d}\left|\int_{[0,1]}l_j(s)l_k(s)\rho(s)\,d\lambda(s)\langle\alpha(u_a,u_b),c\rangle_{\mathbf{C}}\right|^2\\
&=\left\|\langle\alpha(\cdot,\cdot),c\rangle_{\mathbf{C}}\right\|_2^2\cdot\sum_{j,k=1}^{\infty}\int_{[0,1]^2}l_j(s)l_k(s)l_j(t)l_k(t)\rho(s)\bar{\rho}(t)\,d\lambda(s)\,d\lambda(t)\\
&=\left\|\langle\alpha(\cdot,\cdot),c\rangle_{\mathbf{C}}\right\|_2^2\cdot\int_{[0,1]^2}(s\wedge t)^2\rho(s)\bar{\rho}(t)\,d\lambda(s)\,d\lambda(t)\\
&=\left\|\langle\alpha(\cdot,\cdot),c\rangle_{\mathbf{C}}\right\|_2^2\cdot\int_{[0,1]^2}(s\wedge t)^2\,d\eta(s)\,d\bar{\eta}(t),
\end{aligned}\tag{3.40}$$

wherein we have used Eq. (3.39) in the fourth equality above. Summing this equation over c in an orthonormal basis for $\mathbf{C}$ shows

$$\|\omega\|_2^2=\|\alpha\|_2^2\cdot\int_{[0,1]^2}(s\wedge t)^2\,d\eta(s)\,d\bar{\eta}(t).\tag{3.41}$$

3.4. Finite-dimensional projections and cylinder functions

Let $i:H\to W$ be the inclusion map, and $i^*:W^*\to H^*$ be its transpose, i.e. $i^*\ell:=\ell\circ i$ for all $\ell\in W^*$. Also let

$$H_*:=\left\{h\in H\colon\ \langle\cdot,h\rangle_H\in\operatorname{Ran}\left(i^*\right)\subset H^*\right\}$$

or in other words, $h\in H$ is in H_* iff $\langle\cdot,h\rangle_H\in H^*$ extends to a continuous linear functional on W. (We will continue to denote the continuous extension of $\langle\cdot,h\rangle_H$ to W by $\langle\cdot,h\rangle_H$.) Because H is a dense subspace of W, i^* is injective and because i is injective, i^* has a dense range. Since $h\mapsto\langle\cdot,h\rangle_H$ as a map from H to H^* is a conjugate linear isometric isomorphism, it follows from the above comments that $H_*\ni h\to\langle\cdot,h\rangle_H\in W^*$ is a conjugate linear isomorphism too, and that H_* is a dense subspace of H.

Now suppose that $P:H\to H$ is a finite rank orthogonal projection such that $PH\subset H_*$. Let $\{e_j\}_{j=1}^n$ be an orthonormal basis for PH and $\ell_j=\langle\cdot,e_j\rangle_H\in W^*$. Then we may extend P to a (unique) continuous operator from $W\to H$ (still denoted by P) by letting

$$Pw:=\sum_{j=1}^{n}\langle k,e_j\rangle_He_j=\sum_{j=1}^{n}\ell_j(w)e_j\quad\text{for all }w\in W.\tag{3.42}$$

For all $w \in W$ we have, $\|Pw\|_H \leqslant C_2(P)\|Pw\|_W$ and

$$\|Pw\|_W \leqslant \left(\sum_{i=1}^n \big\|\langle \cdot, e_i\rangle_H\big\|_W \|e_i\|_W\right)\|w\|_W$$

and therefore there exists $C < \infty$ such that

$$\|Pw\|_H \leqslant C\|w\|_W \quad \text{for all } w \in W. \tag{3.43}$$

Notation 3.24. Let $\operatorname{Proj}(W)$ denote the collection of finite rank projections on W such that $PW \subset H_*$ and $P|_H : H \to H$ is an orthogonal projection, i.e. P has the form given in Eq. (3.42). Further, let $G_P := PW \times \mathbf{C}$ (a subgroup of G_{CM}) and

$$\pi = \pi_P : G \to G_P$$

be defined by $\pi_P(w, c) := (Pw, c)$.

Remark 3.25. The reader should be aware that $\pi_P : G \to G_P \subset G_{\mathrm{CM}}$ is *not* (for general ω and $P \in \operatorname{Proj}(W)$) a group homomorphism. In fact we have,

$$\pi_P\big[(w,c)\cdot(w',c')\big] - \pi_P(w,c)\cdot\pi_P(w',c') = \Gamma_P(w,w'), \tag{3.44}$$

where

$$\Gamma_P(w,w') = \frac{1}{2}\big(0, \omega(w,w') - \omega(Pw,Pw')\big). \tag{3.45}$$

So unless ω is "supported" on the range of P, π_P is not a group homomorphism. Since, $(w,b) + (0,c) = (w,b)\cdot(0,c)$ for all $w \in W$ and $b, c \in \mathbf{C}$, we may also write Eq. (3.44) as

$$\pi_P\big[(w,c)\cdot(w',c')\big] = \pi_P(w,c)\cdot\pi_P(w',c')\cdot\Gamma_P(w,w'). \tag{3.46}$$

Definition 3.26. A function $f : G \to \mathbb{C}$ is said to be a (*smooth*) *cylinder function* if it may be written as $f = F \circ \pi_P$ for some $P \in \operatorname{Proj}(W)$ and some (smooth) function $F : G_P \to \mathbb{C}$.

Notation 3.27. For $g = (w,c) \in G$, let $\gamma(g)$ and $\chi(g)$ be the elements of $\mathfrak{g}_{\mathrm{CM}} \otimes \mathfrak{g}_{\mathrm{CM}}$ defined by

$$\gamma(g) := \sum_{j=1}^\infty \big(0, \omega(w,e_j)\big) \otimes (e_j, 0) \quad \text{and}$$

$$\chi(g) := \sum_{j=1}^\infty \big(0, \omega(w,e_j)\big) \otimes \big(0, \omega(w,e_j)\big),$$

where $\{e_j\}_{j=1}^\infty$ is any orthonormal basis for H. Both γ and χ are well defined because of Lemma 3.16.

Notation 3.28 *(Left differentials).* Suppose $f : G \to \mathbb{C}$ is a smooth cylinder function. For $g \in G$ and $h, h_1, \dots, h_n \in \mathfrak{g}$, $n \in \mathbb{N}$, let

$$\begin{aligned} \big(D^0 f\big)(g) &= f(g) \quad \text{and} \\ \big(D^n f\big)(g)(h_1 \otimes \dots \otimes h_n) &= \tilde{h}_1 \dots \tilde{h}_n f(g), \end{aligned} \tag{3.47}$$

where $\tilde{h} f$ is given as in Eq. (3.10) or Eq. (3.12). We will write Df for $D^1 f$.

Proposition 3.29. *Let $\{e_j\}_{j=1}^{\infty}$ and $\{f_\ell\}_{\ell=1}^{d}$ be orthonormal bases for H and $\mathbf{C}$, respectively. Then for any smooth cylinder function, $f : G \to \mathbb{C}$,*

$$Lf(g) := \sum_{j=1}^{\infty} \big[\widetilde{(e_j, 0)}^2 f\big](g) + \sum_{\ell=1}^{d} \big[\widetilde{(0, f_\ell)}^2 f\big](g) \tag{3.48}$$

is well defined. Moreover, if $f = F \circ \pi_P$, ∂_h is as in Notation 3.5 for all $h \in \mathfrak{g}_{CM}$,

$$\Delta_H f(g) := \sum_{j=1}^{\infty} \partial^2_{(e_j,0)} f(g) = (\Delta_{PH} F)(Pw, c) \tag{3.49}$$

and

$$\Delta_{\mathbf{C}} f(g) := \sum_{\ell=1}^{d} \big[\partial^2_{(0,f_\ell)} f\big](g) = (\Delta_{\mathbf{C}} F)(Pw, c), \tag{3.50}$$

then

$$Lf(g) = (\Delta_H f + \Delta_{\mathbf{C}} f)(g) + f''(g)\left(\gamma(g) + \frac{1}{4}\chi(g)\right). \tag{3.51}$$

Proof. The proof of the second equality in Eq. (3.49) is straightforward and will be left to the reader. Recall from Eq. (3.12) that

$$\widetilde{(e_j, 0)} f(g) = f'(g)\left(e_j, \frac{1}{2}\omega(w, e_j)\right). \tag{3.52}$$

Applying $\widetilde{(e_j, 0)}$ to both sides of Eq. (3.52) gives

$$\widetilde{(e_j, 0)}^2 f(g) = f''(g)\left(\left(e_j, \frac{1}{2}\omega(w, e_j)\right) \otimes \left(e_j, \frac{1}{2}\omega(w, e_j)\right)\right) \tag{3.53}$$

$$\begin{aligned} &= f''(g)\big((e_j, 0) \otimes (e_j, 0)\big) + f''(g)\big(\big(0, \omega(w, e_j)\big) \otimes (e_j, 0)\big) \\ &\quad + \frac{1}{4} f''(g)\big(\big(0, \omega(w, e_j)\big) \otimes \big(0, \omega(w, e_j)\big)\big), \end{aligned} \tag{3.54}$$

wherein we have used,

$$\partial_{e_j}\omega(\cdot, e_j) = \omega(e_j, e_j) = 0.$$

Summing Eq. (3.54) on j shows,

$$\begin{aligned}\sum_{j=1}^{\infty}\big[\widetilde{(e_j,0)}^2 f\big](g) &= \sum_{j=1}^{\infty} f''(g)\big((e_j,0)\otimes(e_j,0)\big) + f''(g)\left(\gamma(g) + \frac{1}{4}\chi(g)\right)\\ &= \sum_{j=1}^{\infty}\partial^2_{(e_j,0)} f(g) + f''(g)\left(\gamma(g) + \frac{1}{4}\chi(g)\right).\end{aligned}$$

The formula in Eq. (3.51) for Lf is now easily verified and this shows that Lf is independent of the choice of orthonormal bases for H and $\mathbf{C}$ appearing in Eq. (3.48). □

4. Brownian motion and heat kernel measures

For the Hilbert space stochastic calculus background needed for this section, see Métivier [41]. For the background on Itô integral relative to an abstract Wiener space-valued Brownian motion, see Kuo [36, pp. 188–207] (especially Theorem 5.1), Kusuoka and Stroock [37, p. 5], and the appendix in [16].

Suppose now that $(B(t), B_0(t))$ is a smooth curve in $\mathfrak{g}_{CM}$ with $(B(0), B_0(0)) = (0, 0)$ and consider solving, for $g(t) = (w(t), c(t)) \in G_{CM}$, the differential equations

$$\big(\dot{w}(t), \dot{c}(t)\big) = \dot{g}(t) = l_{g(t)*}\big(\dot{B}(t), \dot{B}_0(t)\big) \quad \text{with } g(0) = (0, 0). \tag{4.1}$$

By Eq. (3.11), it follows that

$$\big(\dot{w}(t), \dot{c}(t)\big) = \left(\dot{B}(t), \dot{B}_0(t) + \frac{1}{2}\omega\big(w(t), \dot{B}(t)\big)\right)$$

and therefore the solution to Eq. (4.1) is given by

$$g(t) = \big(w(t), c(t)\big) = \left(B(t), B_0(t) + \frac{1}{2}\int_0^t \omega\big(B(\tau), \dot{B}(\tau)\big)\, d\tau\right). \tag{4.2}$$

Below in Section 4.2, we will replace B and B_0 by Brownian motions and use this to define a Brownian motion on G.

4.1. A quadratic integral

Let $\{(B(t), B_0(t))\}_{t\geqslant 0}$ be a *Brownian motion* on $\mathfrak{g}$ with variance determined by

$$\begin{aligned}&\mathbb{E}\big[\big\langle\big(B(s), B_0(s)\big), (A, a)\big\rangle_{\mathfrak{g}_{CM}}\big\langle\big(B(t), B_0(t)\big), (C, c)\big\rangle_{\mathfrak{g}_{CM}}\big]\\ &\quad = \mathrm{Re}\big\langle(A, a), (C, c)\big\rangle_{\mathfrak{g}_{CM}} \min(s, t)\end{aligned}$$

for all $s, t \in [0, \infty)$, $A, C \in H_*$ and $a, c \in \mathbf{C}$. Also let $\{e_j\}_{j=1}^{\infty} \subset H_*$ be an orthonormal basis for H. For $n \in \mathbb{N}$, define $P_n \in \operatorname{Proj}(W)$ as in Notation 3.24, i.e.

$$P_n(w) = \sum_{j=1}^{n} \langle w, e_j \rangle_H e_j = \sum_{j=1}^{n} \ell_j(w) e_j \quad \text{for all } w \in W. \tag{4.3}$$

Proposition 4.1. *For each n, let $M_t^n := \int_0^t \omega(B(\tau), dP_n B(\tau))$. Then*

(1) *$\{M_t^n\}_{t \geqslant 0}$ is an L^2-martingale and there exists an L^2-martingale, $\{M_t\}_{t \geqslant 0}$ with values in $\mathbf{C}$ such that*

$$\lim_{n \to \infty} \mathbb{E}\Big[\max_{t \leqslant T} \|M_t - M_t^n\|_{\mathbf{C}}^2\Big] = 0 \quad \text{for all } T < \infty. \tag{4.4}$$

(2) *The quadratic variation of M is given by*

$$\langle M \rangle_t = \int_0^t \big\|\omega\big(B(\tau), \cdot\big)\big\|_{H^* \otimes \mathbf{C}}^2 \, d\tau. \tag{4.5}$$

(3) *The square integrable martingale, M_t, is well defined independent of the choice of the orthonormal basis, $\{e_j\}_{j=1}^{\infty}$ and hence will be denoted by $\int_0^t \omega(B(\tau), dB(\tau))$.*

(4) *For each $p \in [1, \infty)$, $\{M_t\}_{t \geqslant 0}$ is L^p-integrable and there exists $c_p < \infty$ such that*

$$\mathbb{E}\Big(\sup_{0 \leqslant t \leqslant T} \|M_t\|_{\mathbf{C}}^p\Big) \leqslant c_p T^p < \infty \quad \text{for all } 0 \leqslant T < \infty.$$

(*This estimate will be considerably generalized in Proposition* 4.13 *below.*)

Proof. (1) For $P \in \operatorname{Proj}(W)$ let $M_t^P := \int_0^t \omega(B(\tau), dPB(\tau))$. Let $P, Q \in \operatorname{Proj}(W)$ and choose an orthonormal basis, $\{v_l\}_{l=1}^N$ for $\operatorname{Ran}(P) + \operatorname{Ran}(Q)$. We then have

$$\begin{aligned} \mathbb{E}\big[\|M_T^P - M_T^Q\|_{\mathbf{C}}^2\big] &= \mathbb{E}\int_0^T \sum_{l=1}^{N} \big\|\omega\big(B(\tau), (P-Q)v_l\big)\big\|_{\mathbf{C}}^2 \, d\tau \\ &= \mathbb{E}\int_0^T \sum_{l=1}^{\infty} \big\|\omega\big(B(\tau), (P-Q)e_l\big)\big\|_{\mathbf{C}}^2 \, d\tau \qquad (4.6) \\ &= \int_0^T \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \big\|\omega\big(e_k, (P-Q)e_l\big)\big\|_{\mathbf{C}}^2 \tau \, d\tau \\ &= \frac{T^2}{2} \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \big\|\omega\big(e_k, (P-Q)e_l\big)\big\|_{\mathbf{C}}^2. \qquad (4.7) \end{aligned}$$

Taking $P = P_n$ and $Q = P_m$ with $m \leqslant n$ in Eq. (4.7) allows us to conclude that

$$\mathbb{E}\big[\big\|M_T^n - M_T^m\big\|_{\mathbf{C}}^2\big] = \frac{T^2}{2}\sum_{j=m+1}^{n}\sum_{l=1}^{\infty}\big\|\omega(e_l, e_j)\big\|_{\mathbf{C}}^2 \to 0 \quad \text{as } m, n \to \infty$$

because $\|\omega\|_2^2 < \infty$ by Proposition 3.14. Since the space of continuous L^2-martingales on $[0, T]$ is complete in the norm, $N \to \mathbb{E}\|N_T\|_{\mathbf{C}}^2$ and, by Doob's maximal inequality [34, Proposition 7.16], there exists $c < \infty$ such that

$$\mathbb{E}\Big[\max_{t\leqslant T}\|N_t\|_{\mathbf{C}}^p\Big] \leqslant c\mathbb{E}\|N_T\|_{\mathbf{C}}^p,$$

it follows that there exists a square integrable $\mathbf{C}$-valued martingale, $\{M_t\}_{t\geqslant 0}$, such that Eq. (4.4) holds.

(2) Since the quadratic variation of M^n is given by

$$\big\langle M^n\big\rangle_t = \int_0^t \big\|\omega\big(B(\tau), dP_nB(\tau)\big)\big\|_{\mathbf{C}}^2 = \int_0^t \sum_{l=1}^{n}\big\|\omega\big(B(\tau), e_l\big)\big\|_{\mathbf{C}}^2\,d\tau$$

and

$$\begin{aligned}\mathbb{E}\big[\big|\langle M\rangle_t - \big\langle M^n\big\rangle_t\big|\big] &\leqslant \sqrt{\mathbb{E}\big[\big\langle M - M^n\big\rangle_t\big]\cdot\mathbb{E}\big[\big\langle M + M^n\big\rangle_t\big]}\\ &= \sqrt{\mathbb{E}\big\|M_t - M_t^n\big\|_{\mathbf{C}}^2\cdot\mathbb{E}\big\|M_t + M_t^n\big\|_{\mathbf{C}}^2} \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

Eq. (4.5) easily follows.

(3) Suppose now that $\{e_j'\}_{j=1}^{\infty} \subset H_*$ is another orthonormal basis for H and $P_n' : W \to H_*$ are the corresponding orthogonal projections. Taking $P = P_n$ and $P' = P_n'$ in Eq. (4.7) gives,

$$\mathbb{E}\big\|M_T^{P_n} - M_T^{P_n'}\big\|_{\mathbf{C}}^2 = \frac{T^2}{2}\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\big\|\omega\big(e_k, \big(P_n - P_n'\big)e_l\big)\big\|_{\mathbf{C}}^2. \tag{4.8}$$

Since

$$\begin{aligned}\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\big\|\omega\big(e_k, P_n'e_l\big) - \omega(e_k, e_l)\big\|_{\mathbf{C}}^2 &= \sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\big\|\omega\big(e_k, \big(P_n' - I\big)e_l\big)\big\|_{\mathbf{C}}^2\\ &= \sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\big\|\omega\big(e_k', \big(P_n' - I\big)e_l'\big)\big\|_{\mathbf{C}}^2\\ &= \sum_{l=n+1}^{\infty}\sum_{k=1}^{\infty}\big\|\omega\big(e_k', e_l'\big)\big\|_{\mathbf{C}}^2 \to 0 \quad \text{as } n \to \infty\end{aligned}$$

and similarly but more easily, $\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\|\omega(e_k,P_ne_l)-\omega(e_k,e_l)\|_{\mathbf{C}}^2\to 0$ as $n\to\infty$, we may pass to the limit in Eq. (4.8) to learn that $\lim_{n\to\infty}\mathbb{E}\|M_T^{P_n}-M_T^{P_n'}\|_{\mathbf{C}}^2=0$.

(4) By Jensen's inequality

$$\left(\int_0^T \|\omega(B(s),\cdot)\|_{H^*\otimes\mathbf{C}}^2\,ds\right)^{p/2} = T^{p/2}\left(\int_0^T \|\omega(B(s),\cdot)\|_{H^*\otimes\mathbf{C}}^2\,\frac{ds}{T}\right)^{p/2}$$
$$\leqslant T^{p/2}\int_0^T \|\omega(B(s),\cdot)\|_{H^*\otimes\mathbf{C}}^p\,\frac{ds}{T}$$
$$= T^{\frac{p}{2}-1}\int_0^T \|\omega(B(s),\cdot)\|_{H^*\otimes\mathbf{C}}^p\,ds.$$

Combining this estimate with Eq. (3.28) and then applying either Skorohod's or Fernique's inequality (see Eqs. (2.3) or (2.4)) shows

$$\begin{aligned}\mathbb{E}\big[\langle M\rangle_T^{p/2}\big] &\leqslant T^{\frac{p}{2}-1}\int_0^T \mathbb{E}\|\omega(B(s),\cdot)\|_{H^*\otimes\mathbf{C}}^p\,ds\\ &\leqslant T^{\frac{p}{2}-1}\int_0^T C_2^{p/2}\|\omega\|_0^p\|B(s)\|_W^p\,ds\\ &\leqslant T^{\frac{p}{2}-1}C_2^{p/2}\|\omega\|_0^p\int_W \|y\|_W^p\,d\mu(y)\int_0^T s^{p/2}\,ds\\ &= T^{\frac{p}{2}-1}C_2^{p/2}\|\omega\|_0^p C_p\frac{T^{p/2+1}}{p/2+1} = c_p' T^p. \end{aligned} \tag{4.9}$$

As a consequence of the Burkholder–Davis–Gundy inequality (see for example [49, Corollary 6.3.1a, p. 344], [45, Appendix A.2], or [41, p. 212] and [34, Theorem 17.7] for the real case), for any $p\geqslant 2$ there exists $c_p''<\infty$ such that

$$\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}\|M_t\|_{\mathbf{C}}\Big)^p \leqslant c_p''\mathbb{E}\big[\langle M\rangle_T^{p/2}\big] = c_p''c_p'T^p =: c_pT^p. \qquad \square$$

4.2. Brownian motion on $G(\omega)$

Motivated by Eq. (4.2) we have the following definition.

Definition 4.2. Let $(B(t), B_0(t))$ be a $\mathfrak{g}$-valued Brownian motion as in Section 4.1. A *Brownian motion* on G is the continuous G-valued process defined by

$$g(t) = \left(B(t), B_0(t) + \frac{1}{2}\int_0^t \omega(B(\tau), dB(\tau))\right). \tag{4.10}$$

Further, for $T > 0$, let $\nu_T = \mathrm{Law}(g(T))$ be a probability measure on G. We refer to ν_T as the *time T heat kernel measure on G*.

Remark 4.3. An alert reader may complain that we should use the Stratonovich integral in Eq. (4.10) rather than the Itô integral. However, these two integrals are equal since ω is a skew symmetric form

$$\int_0^t \omega\big(B(\tau), \circ dB(\tau)\big) = \int_0^t \omega\big(B(\tau), dB(\tau)\big) + \frac{1}{2}\int_0^t \omega\big(dB(\tau), dB(\tau)\big)$$
$$= \int_0^t \omega\big(B(\tau), dB(\tau)\big).$$

Theorem 4.4 *(The generator of $g(t)$). The generator of $g(t)$ is the operator L defined in Proposition* 3.29. *More precisely, if $f : G \to \mathbb{C}$ is a smooth cylinder function, then*

$$d\big[f\big(g(t)\big)\big] = f'\big(g(t)\big)\,dg(t) + \frac{1}{2}Lf\big(g(t)\big)\,dt, \tag{4.11}$$

where L is given in Proposition 3.29, *f' is defined as in Notation* 3.5 *and*

$$dg(t) = \left(dB(t), dB_0(t) + \frac{1}{2}\omega\big(B(t), dB(t)\big)\right).$$

Proof. Let us begin by observing that

$$dg(t) \otimes dg(t) = \left(dB(t), dB_0(t) + \frac{1}{2}\omega\big(B(t), dB(t)\big)\right)^{\otimes 2}$$
$$= \left[\left(dB(t), \frac{1}{2}\omega\big(B(t), dB(t)\big)\right) + \big(0, dB_0(t)\big)\right]^{\otimes 2}$$
$$= \sum_{j=1}^{\infty}\left(e_j, \frac{1}{2}\omega\big(B(t), e_j\big)\right)^{\otimes 2} dt + \sum_{\ell=1}^{d}(0, f_\ell)^{\otimes 2}\,dt, \tag{4.12}$$

where $\{f_\ell\}_{\ell=1}^d$ is an orthonormal basis for $\mathbf{C}$ and $\{e_j\}_{j=1}^\infty$ is an orthonormal basis for H. Hence, as a consequence of Itô's formula, we have

$$d\big[f\big(g(t)\big)\big] = f'\big(g(t)\big)\,dg(t) + \frac{1}{2}f''\big(g(t)\big)\big(dg(t)\otimes dg(t)\big)$$
$$= f'\big(g(t)\big)\,dg(t) + \frac{1}{2}f''\big(g(t)\big)\sum_{j=1}^{\infty}\left(e_j, \frac{1}{2}\omega\big(B(t), e_j\big)\right)^{\otimes 2} dt$$
$$+ \frac{1}{2}f''\big(g(t)\big)\sum_{\ell=1}^{d}(0, f_\ell)^{\otimes 2}\,dt$$

$$= f'\big(g(t)\big)\,dg(t) + \frac{1}{2}\sum_{j=1}^{\infty}\big(\widetilde{(e_j,0)}^2 f\big)\big(g(t)\big)\,dt + \frac{1}{2}\sum_{\ell=1}^{d}\big(\widetilde{(0,f_\ell)}^2 f\big)\big(g(t)\big)\,dt$$

$$= f'\big(g(t)\big)\big(dg(t)\big) + \frac{1}{2}Lf\big(g(t)\big)\,dt. \qquad \square$$

For the next corollary, let $P \in \mathrm{Proj}(W)$ as in Eq. (3.42), $F \in C^2(PH \times \mathbf{C}, \mathbb{C})$, and $f = F \circ \pi_P : G \to \mathbb{C}$ be a cylinder function where $P \in \mathrm{Proj}(W)$. We will further suppose there exist $0 < K, p < \infty$ such that

$$\big|F(h,c)\big| + \big\|F'(h,c)\big\| + \big\|F''(h,c)\big\| \leqslant K\big(1 + \|h\|_H + \|c\|_{\mathbf{C}}\big)^p \tag{4.13}$$

for any $h \in PH$ and $c \in \mathbf{C}$. Further let $\{f_\ell\}_{\ell=1}^d$ be an orthonormal basis for $\mathbf{C}$ and extend $\{e_j\}_{j=1}^n$ to an orthonormal basis, $\{e_j\}_{j=1}^\infty$, for H.

Corollary 4.5. *If $f : G \to \mathbb{C}$ is a cylinder function as above, then*

$$\mathbb{E}\big[f\big(g(T)\big)\big] = f(\mathbf{e}) + \frac{1}{2}\int_0^T \mathbb{E}\big[(Lf)\big(g(t)\big)\big]\,dt, \tag{4.14}$$

i.e.

$$\nu_T(f) = f(\mathbf{e}) + \frac{1}{2}\int_0^T \nu_t(Lf)\,dt. \tag{4.15}$$

In other words, ν_t weakly solves the heat equation

$$\partial_t \nu_t = \frac{1}{2}L\nu_t \quad \text{with } \lim_{t\downarrow 0}\nu_t = \delta_{\mathbf{e}}.$$

Proof. Integrating Eq. (4.11) shows

$$f\big(g(T)\big) = f(\mathbf{e}) + N_T + \frac{1}{2}\int_0^T Lf\big(g(\tau)\big)\,d\tau, \tag{4.16}$$

where

$$N_t := \int_0^t f'\big(g(\tau)\big)\,dg(\tau) = \int_0^t f'\big(g(\tau)\big)\left(dB(\tau), dB_0(\tau) + \frac{1}{2}\,dM_\tau\right)$$

and $M_t = \int_0^t \omega(B(\tau), dB(\tau))$. Using Eqs. (4.12) and (4.13) there exists $C = C(P, \|\omega_0\|) < \infty$ such that

$$\begin{aligned} d\langle N\rangle_t &:= |dN_t|^2 = \langle f'(g_t)\otimes f'(g_t), dg_t\otimes dg_t\rangle \\ &= \sum_{j=1}^{\infty}\left| f'\big(g(t)\big)\left(e_j, \frac{1}{2}\omega\big(B(t), e_j\big)\right)\right|^2 dt + \sum_{\ell=1}^{d}\left|f'\big(g(t)\big)(0, f_\ell)\right|^2 dt \\ &= \sum_{j=1}^{n}\left| f'\big(g(t)\big)\left(e_j, \frac{1}{2}\omega\big(B(t), e_j\big)\right)\right|^2 dt + \sum_{\ell=1}^{d}\left|f'\big(g(t)\big)(0, f_\ell)\right|^2 dt \\ &\leqslant C_1\big(P, \|\omega_0\|\big)\big(1 + \|PB(t)\|_H + \|B_0(t)\|_{\mathbf{C}}\big)^p\big(\|B(t)\|_W^2 + 1\big)\, dt \\ &\leqslant C\big(1 + \|B(t)\|_W + \|B_0(t)\|_{\mathbf{C}}\big)^{p+2}\, dt, \end{aligned}$$

wherein we have used Eq. (3.43) for the last inequality. From this inequality and either of Eqs. (2.3) or (2.4), we find

$$\mathbb{E}\big[\langle N\rangle_T\big] \leqslant C\int_0^T \mathbb{E}\big(1 + \|B(t)\|_W + \|B_0(t)\|_{\mathbf{C}}\big)^{p+2}\, dt < \infty$$

and hence that N_t is a square integrable martingale. Therefore we may take the expectation of Eq. (4.16) which implies Eq. (4.14). □

4.3. Finite-dimensional approximations

Proposition 4.6. *Let* $\{P_n\}_{n=1}^{\infty} \subset \operatorname{Proj}(W)$ *be as in Eq.* (4.3) *and*

$$B_n(t) := P_n B(t) \in P_n H \subset H \subset W. \tag{4.17}$$

Then

$$\lim_{n\to\infty} \mathbb{E}\Big[\max_{0\leqslant t\leqslant T}\|B(t) - B_n(t)\|_W^p\Big] = 0 \quad \textit{for all } p \in [1,\infty), \tag{4.18}$$

and

$$\lim_{n\to\infty}\max_{0\leqslant t\leqslant T}\|B(t) - B_n(t)\|_W = 0 \quad \textit{a.s.} \tag{4.19}$$

Proof. Let $\{w_k\}_{k=1}^{\infty} \subset W$ be a countable dense set and for each $k \in \mathbb{N}$, choose $\varphi_k \in W^*$ such that $\|\varphi_k\|_{W^*} = 1$ and $\varphi_k(w_k) = \|w_k\|_W$. We then have,

$$\|w\|_W = \sup_k\big|\varphi_k(w)\big| = \sup \operatorname{Re}\varphi_k(w) \quad \text{for all } w \in W.$$

By [8, Theorem 3.5.7] with $A = I$, if $\varepsilon_n(t) := B(t) - B_n(t)$, then

$$\lim_{n\to\infty}\mathbb{E}\|\varepsilon_n(T)\|_W^p = 0 \quad \text{for all } p \in [1,\infty). \tag{4.20}$$

Since $\{\varphi_k(\varepsilon_n(t))\}_{t\geqslant 0}$ is (up to a multiplicative factor) a standard Brownian motion, we have $\{|\varphi_k(\varepsilon_n(t))|\}_{t\geqslant 0}$ is a submartingale for each $k \in \mathbb{N}$ and therefore so is

$$\left\{\left\|\varepsilon_n(t)\right\| = \sup_k\left|\varphi_k(\varepsilon_n(t))\right|\right\}_{t\geqslant 0}.$$

Hence, according to Doob's inequality, for each $p \in [1,\infty)$ there exists $C_p < \infty$ such that

$$\mathbb{E}\left|\max_{t\leqslant T}\left\|\varepsilon_n(t)\right\|_W\right|^p \leqslant C_p\mathbb{E}\left\|\varepsilon_n(T)\right\|_W^p. \tag{4.21}$$

Combining Eq. (4.21) with Eq. (4.20) proves Eq. (4.18). Eq. (4.19) now follows from Eq. (4.18) and [11, Proposition 2.11]. To apply this proposition, let E be the Banach space, $C([0,T],W)$ equipped with the sup-norm, and let $\xi_k := \ell_k(B(\cdot))e_k \in E$ for all $k \in \mathbb{N}$. □

Lemma 4.7 *(Finite-dimensional approximations to $g(t)$). For $P \in \operatorname{Proj}(W)$, $Q := I_W - P$, let $g_P(t)$ be the Brownian motion on G_P defined by*

$$g_P(t) := \left(PB(t), B_0(t) + \frac{1}{2}\int_0^t \omega\big(PB(\tau), P\,dB(\tau)\big)\right).$$

Then

$$g(t) = g_P(t)\left(QB(t), \frac{1}{2}\int_0^t \big[2\omega\big(QB(\tau), P\,dB(\tau)\big) + \omega\big(QB(\tau), Q\,dB(\tau)\big)\big]\right), \tag{4.22}$$

and

$$g_P(t)^{-1}\pi_P\big(g(t)\big) = \frac{1}{2}\left(0, \int_0^t \big[\omega\big(B(\tau), dB(\tau)\big) - \omega\big(PB(\tau), P\,dB(\tau)\big)\big]\right). \tag{4.23}$$

Also, if $\{P_n\}_{n=1}^\infty \subset \operatorname{Proj}(W)$ are as in Eq. (4.3) and

$$g_n(t) = g_{P_n}(t) = \left(P_nB(t), B_0(t) + \frac{1}{2}\int_0^t \omega\big(P_nB(\tau), dP_nB(\tau)\big)\right), \tag{4.24}$$

then

$$\lim_{n\to\infty}\mathbb{E}\Big[\max_{0\leqslant t\leqslant T}\left\|g(t) - g_n(t)\right\|_{\mathfrak{g}}^p\Big] = 0 \tag{4.25}$$

for all $1 \leqslant p < \infty$.

Proof. A simple computation shows

$$l_{g_P(t)^{-1}*} \circ dg_P(t) = \begin{pmatrix} dPB(t), dB_0(t) + \frac{1}{2}\omega(PB(t), P\,dB(t)) \\ + \frac{1}{2}\omega(-PB(t), P\,dB(t)) \end{pmatrix}$$
$$= \big(dPB(t), dB_0(t)\big) = d\big(PB(t), B_0(t)\big).$$

Hence it follows that g_P solves the stochastic differential equation,

$$dg_P(t) = l_{g_P(t)*} \circ d\big(PB(t), B_0(t)\big) \quad \text{with } g_P(0) = 0$$

and therefore g_P is a G_P-valued Brownian motion. The proof of the equalities in Eqs. (4.22) and (4.23) follows by elementary manipulations which are left to the reader.

In light of Eq. (4.18) of Proposition 4.6, to prove the last assertion we must show

$$\lim_{n\to\infty} \mathbb{E}\Big[\max_{0\leqslant t\leqslant T} \big|M_t(n)\big|^p\Big] = 0, \tag{4.26}$$

where $M_t(n)$ is the local martingale defined by

$$M_t(n) := \int_0^t \big[\omega\big(B(\tau), dB(\tau)\big) - \omega\big(B_n(\tau), dB_n(\tau)\big)\big].$$

Since

$$\big\langle M(n)\big\rangle_T = \sum_{j=1}^{\infty} \int_0^T \big\|\omega\big(B(\tau), e_j\big)\big\|_{\mathbf{C}}^2\, d\tau + \sum_{j=1}^{n} \int_0^T \big\|\omega\big(B_n(\tau), e_j\big)\big\|_{\mathbf{C}}^2\, d\tau$$
$$- 2\sum_{j=1}^{n} \int_0^T \big\langle \omega\big(B(\tau), e_j\big), \omega\big(B_n(\tau), e_j\big)\big\rangle_{\mathbf{C}}\, d\tau$$

and

$$\frac{2}{T^2}\mathbb{E}\big[\big\langle M(n)\big\rangle_T\big] = \sum_{j,k=1}^{\infty} \big\|\omega(e_k, e_j)\big\|_{\mathbf{C}}^2 - \sum_{k=1}^{n}\sum_{j=1}^{n} \big\|\omega(e_k, e_j)\big\|_{\mathbf{C}}^2 \to 0$$

as $n \to \infty$, it follows by the Burkholder–Davis–Gundy inequalities that $M(n)$ is a martingale and Eq. (4.26) holds for $p = 2$ and hence for $p \in [1, 2]$.

By Doob's maximal inequality [34, Proposition 7.16], to prove Eq. (4.26) for $p \geqslant 2$, it suffices to show $\lim_{n\to\infty} \mathbb{E}[|M_T(n)|^p] = 0$. However, $M_T(n)$ has Itô's chaos expansion terminating at degree two and hence by a theorem of Nelson (see [44, Lemma 2, p. 415] and [43, pp. 216–217]) for each $j \in \mathbb{N}$ there exists $c_j < \infty$ such that $\mathbb{E}[M_T^{2j}(n)] \leqslant c_j[\mathbb{E}M_T^2(n)]^j$. (This result also follows from Nelson's hypercontractivity for the Ornstein–Uhlenbeck operator.) This clearly suffices to complete the proof of the theorem. □

Lemma 4.8. *For all $P \in \operatorname{Proj}(W)$ and $t > 0$, let $\nu_t^P := \operatorname{Law}(g_P(t))$. Then $\nu_t^P(dx) = p_t^P(e,x)\,dx$, where dx is the Riemannian volume measure (equal to a Haar measure) $p_t^P(x,y)$ is the heat kernel on G_P.*

Proof. An application of Corollary 4.5 with G replaced by G_P implies that $\nu_t^P = \operatorname{Law}(g_P(t))$ is a weak solution to the heat equation on G_P. The result now follows as an application of [17, Theorem 2.6]. □

Corollary 4.9. *For any $T > 0$, the heat kernel measure ν_T is invariant under the inversion map, $g \mapsto g^{-1}$ for any $g \in G$.*

Proof. It is well known (see for example [20, Proposition 3.1]) that heat kernel measures based at the identity of a finite-dimensional Lie group are invariant under inversion. Now suppose that $f : G \to \mathbb{R}$ is a bounded continuous function. By passing to a subsequence if necessary, we may assume that the sequence of G-valued random variables, $\{g_n(T)\}_{t\geqslant 0}$, in Lemma 4.7 converges almost surely to $g(T)$. Therefore by the dominated convergence theorem,

$$\mathbb{E}f\big(g(T)^{-1}\big) = \lim_{n\to\infty} \mathbb{E}f\big(g_n(T)^{-1}\big) = \lim_{n\to\infty} \mathbb{E}f\big(g_n(T)\big) = \mathbb{E}f\big(g(T)\big).$$

This completes the proof because ν_T is the law of $g(T)$. □

We are now going to give exponential bounds which are much stronger than the moment estimates in Eq. (4.9) of Proposition 4.1. Before doing so we need to recall the following result of Cameron, Martin and Kac [9,33].

Lemma 4.10 *(Cameron–Martin and Kac). Let $\{b_s\}_{s\geqslant 0}$ be a one-dimensional Brownian motion. Then for any $T > 0$ and $\lambda \in [0, \frac{\pi}{2T})$*

$$\mathbb{E}\left[\exp\left(\frac{\lambda^2}{2}\int_0^T b_s^2\,ds\right)\right] = \big[\cos(\lambda T)\big]^{-1/2} < \infty. \tag{4.27}$$

Proof. When $T = 1$, simply follow the proof of [31, Eq. (6.9), p. 472] with λ replaced by $-\lambda^2$. For general $T > 0$, by a change of variables and a Brownian motion scaling we have

$$\int_0^T b_s^2\,ds = T\int_0^1 b_{tT}^2\,dt \stackrel{d}{=} T^2\int_0^1 b_t^2\,dt.$$

Therefore,

$$\begin{aligned}\mathbb{E}\left[\exp\left(\frac{\lambda^2}{2}\int_0^T b_s^2\,ds\right)\right] &= \mathbb{E}\left[\exp\left(\frac{\lambda^2 T^2}{2}\int_0^1 b_s^2\,ds\right)\right]\\ &= \cos^{-1/2}\big(\sqrt{\lambda^2 T^2}\big)\end{aligned} \tag{4.28}$$

provided that $\lambda \in [0, \frac{\pi}{2T})$. □

Remark 4.11. For our purposes below, all we really need later from Lemma 4.10 is the qualitative statement that for $\lambda T > 0$ sufficiently small

$$\mathbb{E}\left[\exp\left(\frac{\lambda^2}{2}\int_0^T b_s^2\,ds\right)\right] = 1 + \frac{\lambda^2 T^2}{4} + O\left(\lambda^4 T^4\right). \tag{4.29}$$

Instead of using Lemma 4.10 we can derive this statement as an easy consequence of the scaling identity in Eq. (4.28) along with the analyticity (use Fernique's theorem) of the function,

$$F(z) := \mathbb{E}\left[\exp\left(z\int_0^1 b_s^2\,ds\right)\right] \quad \text{for } |z| \text{ small.}$$

Proposition 4.12. *If $\{N_t\}_{t\geqslant 0}$ is a continuous local martingale such that $N_0 = 0$, then*

$$\mathbb{E}e^{|N_t|} \leqslant 2\sqrt{\mathbb{E}\left[e^{2\langle N\rangle_t}\right]}. \tag{4.30}$$

Proof. By Itô's formula, we know that

$$Z_t := e^{2N_t - \langle 2N\rangle_t/2} = e^{2N_t - 2\langle N\rangle_t}$$

is a non-negative local martingale. If $\{\sigma_n\}_{n=1}^\infty$ is a localizing sequence of stopping times for Z, then, by Fatou's lemma,

$$\mathbb{E}[Z_t|\mathcal{B}_s] \leqslant \liminf_{n\to\infty}\mathbb{E}\left[Z_t^{\sigma_n}|\mathcal{B}_s\right] = \liminf_{n\to\infty} Z_s^{\sigma_n} = Z_s.$$

This shows that Z is a supermartingale and in particular that $\mathbb{E}[Z_t] \leqslant \mathbb{E}Z_0 = 1$.

By the Cauchy–Schwarz inequality we find

$$\begin{aligned}\mathbb{E}\left[e^{N_t}\right] &= \mathbb{E}\left[e^{N_t - \langle N\rangle_t}e^{\langle N\rangle_t}\right] \\ &\leqslant \sqrt{\mathbb{E}\left[e^{2N_t - 2\langle N\rangle_t}\right]\cdot\mathbb{E}\left[e^{2\langle N\rangle_t}\right]} = \sqrt{\mathbb{E}[Z_t]\cdot\mathbb{E}\left[e^{2\langle N\rangle_t}\right]} \\ &\leqslant \sqrt{\mathbb{E}\left[e^{2\langle N\rangle_t}\right]}. \end{aligned} \tag{4.31}$$

Applying this inequality with N replaced by $-N$ and using $e^{|x|} \leqslant e^x + e^{-x}$ easily give Eq. (4.30). □

Proposition 4.13. *Let $n \in \mathbb{N}$, $T > 0$, $d = \dim_{\mathbb{R}} \mathbf{C}$,*

$$\gamma := \sup\left\{\sum_{j=1}^\infty \left|\left\langle\omega(h, e_j), c\right\rangle_{\mathbf{C}}\right|^2\colon\ \|h\|_H = \|c\|_{\mathbf{C}} = 1\right\} \leqslant \|\omega\|_2^2 < \infty \tag{4.32}$$

and for $P \in \operatorname{Proj}(W)$ let $B_P(t) := PB(t)$. Then for all

$$0 \leqslant \lambda < \frac{\pi}{4dT\sqrt{\gamma}}, \tag{4.33}$$

$$\sup_{P\in\mathrm{Proj}(W)} \mathbb{E}\left[\exp\left(\lambda\left\|\int_0^t \omega\big(B_P(\tau), dB_P(\tau)\big)\right\|_{\mathbf{C}}\right)\right] < \infty \tag{4.34}$$

and

$$\mathbb{E}\left[\exp\left(\lambda\left\|\int_0^t \omega\big(B(\tau), dB(\tau)\big)\right\|_{\mathbf{C}}\right)\right] < \infty. \tag{4.35}$$

Proof. Eq. (4.35) follows by choosing $\{P_n\}_{n=1}^\infty \subset \mathrm{Proj}(W)$ as in Eq. (4.3) and then using Fatou's lemma in conjunction with the estimate in Eq. (4.34). So we need only to concentrate on proving Eq. (4.34).

Fix a $P \in \mathrm{Proj}(W)$ as in Eq. (3.42) and let

$$M_t^P := \int_0^t \omega\big(B_P(\tau), dB_P(\tau)\big).$$

If $\{f_\ell\}_{\ell=1}^d$ is an orthonormal basis for $\mathbf{C}$, then

$$\|M_t^P\|_{\mathbf{C}} \leqslant \sum_{\ell=1}^d \big|\langle M_t^P, f_\ell\rangle_{\mathbf{C}}\big|,$$

and it follows by Hölder's inequality and the martingale estimate in Proposition 4.12 that

$$\begin{aligned}
\mathbb{E}\big[e^{\lambda\|M_t^P\|_{\mathbf{C}}}\big] &\leqslant \mathbb{E}\big[e^{\lambda\sum_{\ell=1}^d |\langle M_t^P, f_\ell\rangle_{\mathbf{C}}|}\big] \leqslant \prod_{\ell=1}^d \big(\mathbb{E}\big[e^{\lambda d|\langle M_t^P, f_\ell\rangle_{\mathbf{C}}|}\big]\big)^{1/d} \\
&\leqslant \prod_{\ell=1}^d \big(2\sqrt{\mathbb{E}\big[e^{2\lambda^2 d^2\langle\langle M_\cdot^P, f_\ell\rangle_{\mathbf{C}}\rangle_t}\big]}\big)^{1/d} \\
&= 2\prod_{\ell=1}^d \big(\mathbb{E}\big[e^{2\lambda^2 d^2\langle\langle M_\cdot^P, f_\ell\rangle_{\mathbf{C}}\rangle_t}\big]\big)^{1/2d}.
\end{aligned} \tag{4.36}$$

We will now evaluate each term in the product in Eq. (4.36). So let $c := f_\ell$ and $N_t := \langle M_t^P, c\rangle_{\mathbf{C}}$, and $Q_P : H \to H$ and $Q : H \to H$ be the unique non-negative symmetric operators such that, for all $h \in H$,

$$\sum_{j=1}^n \big|\big\langle\omega(Ph, e_j), c\big\rangle_{\mathbf{C}}\big|^2 = \langle Q_P h, h\rangle_H \quad \text{for all } h \in H$$

and

$$\sum_{j=1}^{\infty}\left|\left\langle\omega(h,e_j),c\right\rangle_{\mathbf{C}}\right|^2=\langle Qh,h\rangle_H\quad\text{for all } h\in H.$$

Also let $\{q_l(P)\}_{l=1}^{\infty}$ be the eigenvalues listed in decreasing order (counted with multiplicities) for Q_P and observe that

$$q_1(P)=\sup_{h\neq 0}\frac{\langle Q_P h,h\rangle}{\|h\|_H^2}\leqslant\sup_{h\neq 0}\frac{\langle QPh,Ph\rangle}{\|h\|_H^2}\leqslant\sup_{h\neq 0}\frac{\langle Qh,h\rangle}{\|h\|_H^2}\leqslant\gamma. \tag{4.37}$$

With this notation, the quadratic variation of N is given by

$$\langle N\rangle_T=\int_0^T\sum_{j=1}^{n}\left|\left\langle\omega\left(B_P(t),e_j\right),c\right\rangle_{\mathbf{C}}\right|^2dt=\int_0^T\left\langle Q_P B_P(t),B_P(t)\right\rangle_H dt. \tag{4.38}$$

Moreover, by expanding $B_P(\tau)$ in an orthonormal basis of eigenvectors of $Q_P|_{PH}$ it follows that

$$\langle N\rangle_T=\sum_{l=1}^{n}q_l(P)\int_0^T b_l^2(\tau)\,d\tau, \tag{4.39}$$

where $\{b_l\}_{l=1}^{n}$ is a sequence of independent Brownian motions. Hence it follows that

$$\begin{aligned}\mathbb{E}\left[e^{2\lambda^2d^2\langle\langle M^P_\cdot,f_\ell\rangle_{\mathbf{C}}\rangle_T}\right]&=\mathbb{E}\left[e^{2\lambda^2d^2\langle N\rangle_T}\right]\\&=\prod_{l=1}^{n}\mathbb{E}\left[\exp\left(2\lambda^2d^2q_l(P)\int_0^T b_l^2(\tau)\,d\tau\right)\right].\end{aligned} \tag{4.40}$$

If Eq. (4.33) holds then (using Eq. (4.37))

$$2\lambda d\sqrt{q_1(P)}=\sqrt{4\lambda^2d^2q_1(P)}\leqslant 2\lambda d\sqrt{\gamma}<\pi/2T$$

and we may apply Lemma 4.10 to find

$$\begin{aligned}\mathbb{E}\left[\exp\left(2\lambda^2d^2q_l(P)\int_0^T b_l^2(\tau)\,d\tau\right)\right]&=\frac{1}{\sqrt{\cos(2\lambda d\sqrt{q_l(P)}T)}}\\&=\exp\left(-\frac{1}{2}\ln\cos\left(2\lambda d\sqrt{q_l(P)}T\right)\right).\end{aligned} \tag{4.41}$$

Moreover, a simple calculus exercise shows for any $k\in(0,\pi/2)$ there exists $c(k)<\infty$ such that $-\frac{1}{2}\ln\cos(x)\leqslant c(k)x^2$ for $0\leqslant x\leqslant k$. Taking $k=2\lambda d\sqrt{\gamma}T$ we may apply this estimate to

Eq. (4.41) and combine the result with Eq. (4.40) to find

$$\mathbb{E}\left[e^{2\lambda^2 d^2 \langle\langle M_\cdot^P, f_\ell\rangle_{\mathbf{C}}\rangle_T}\right] \leqslant \prod_{l=1}^{n} \exp\left(c(k)4\lambda^2 d^2 T^2 q_l(P)\right) = \exp\left(c(k)4\lambda^2 d^2 T^2 \operatorname{tr}(Q_P)\right).$$

Since $Q_P \leqslant PQP \leqslant Q$, we have

$$\operatorname{tr} Q_P \leqslant \operatorname{tr} Q = \sum_{l=1}^{\infty} \langle Q e_l, e_l\rangle_H = \sum_{j,l=1}^{\infty} \left|\langle \omega(e_l, e_j), c\rangle_{\mathbf{C}}\right|^2 = \left\|\langle \omega(\cdot,\cdot), c\rangle_{\mathbf{C}}\right\|_2^2 < \infty.$$

Combining the last two equations (recalling that $c = f_\ell$) then shows,

$$\mathbb{E}\left[e^{2\lambda^2 d^2 \langle\langle M_\cdot^P, f_\ell\rangle_{\mathbf{C}}\rangle_T}\right] \leqslant \exp\left(c(k)4\lambda^2 d^2 T^2 \left\|\langle \omega(\cdot,\cdot), f_\ell\rangle_{\mathbf{C}}\right\|_2^2\right). \tag{4.42}$$

Using this estimate back in Eq. (4.36) gives,

$$\begin{aligned}\mathbb{E}\left[e^{\lambda\|M_t^P\|_{\mathbf{C}}}\right] &\leqslant 2\exp\left(c(k)2\lambda^2 d T^2 \sum_{\ell=1}^{d} \left\|\langle \omega(\cdot,\cdot), f_\ell\rangle_{\mathbf{C}}\right\|_2^2\right)\\ &= 2\exp\left(c(k)2\lambda^2 d T^2 \|\omega\|_2^2\right)\end{aligned} \tag{4.43}$$

which completes the proof as this last estimate is independent of $P \in \operatorname{Proj}(W)$. □

Proposition 4.14. *Suppose that ν and μ are Gaussian measures on W such $q_\nu(f) := \nu(f^2) \leqslant q_\mu(f) := \mu(f^2)$ for all $f \in W_{\mathbb{R}}^*$. If $g : [0,\infty) \to [0,\infty)$ is a non-negative, non-decreasing, C^1-function, then*

$$\int_W g\left(\|w\|\right) d\nu(w) \leqslant \int_W g\left(\|w\|\right) d\mu(w).$$

Proof. Theorem 3.3.6 in [8, p. 107] states that if $q_\nu \leqslant q_\mu$ then $\mu(A) \leqslant \nu(A)$ for every Borel set A which is convex and balanced. In particular, since $B_t := \{w \in W\colon \|w\| < t\}$ is convex and balanced, it follows that $\mu(B_t) \leqslant \nu(B_t)$ or equivalently that $1 - \nu(B_t) \leqslant 1 - \mu(B_t)$ for all $t \geqslant 0$. Since

$$\begin{aligned}\int_W g\left(\|w\|\right) d\nu(w) &= \int_W \left[g(0) + \int_0^\infty 1_{t \leqslant \|w\|} g'(t)\, dt\right] d\nu(w)\\ &= g(0) + \int_0^\infty \left(g'(t) \int_W 1_{t \leqslant \|w\|}\, d\nu(w)\right) dt\\ &= g(0) + \int_0^\infty g'(t)\left[1 - \nu(B_t)\right] dt\end{aligned} \tag{4.44}$$

with the same formula holding when ν is replaced by μ, it follows that

$$\int_W g(\|w\|)\,d\nu(w) = g(0) + \int_0^\infty g'(t)\big[1-\nu(B_t)\big]\,dt$$
$$\leqslant g(0) + \int_0^\infty dt\, g'(t)\big[1-\mu(B_t)\big] = \int_W g(\|w\|)\,d\mu(w). \qquad \square$$

Definition 4.15. Let $\rho^2 : G \to [0,\infty)$ be defined as

$$\rho^2(w,c) := \|w\|_W^2 + \|c\|_{\mathbf{C}}.$$

In analogy to Gross' theory of measurable semi-norms (see e.g. Definition 5 in [27]) in the abstract Wiener space setting and in light of Theorem 3.12, we view ρ as a "measurable" extension of $d_{G_{\mathrm{CM}}}$.

Theorem 4.16 *(Integrated Gaussian heat kernel bounds). There exists a $\delta > 0$ such that for all $\varepsilon \in (0,\delta)$, $T > 0$, $p \in [1,\infty)$,*

$$\sup_{P\in\mathrm{Proj}(W)} \mathbb{E}\big[e^{\frac{\varepsilon}{T}\rho^2(g_P(T))}\big] < \infty \quad \textit{and} \quad \int_G e^{\frac{\varepsilon}{T}\rho^2(g)}\,d\nu_T(g) < \infty \tag{4.45}$$

whenever $\varepsilon < \delta$.

Proof. Let $\varepsilon' := \varepsilon/T$. For $P \in \mathrm{Proj}(W)$,

$$\rho^2\big(g_P(T)\big) \leqslant \big\|B_P(T)\big\|_W^2 + \big\|B_0(T)\big\|_{\mathbf{C}} + \frac{1}{2}\big\|N_P(T)\big\|_{\mathbf{C}},$$

where $N_P(T) := \int_0^T \omega(B_P(t), dB_P(t))$ and therefore,

$$\mathbb{E}\big[e^{\varepsilon'\rho^2(g_P(T))}\big] \leqslant \mathbb{E}\big[e^{\varepsilon'[\|B_P(T)\|_W^2 + \frac{1}{2}\|N_P(T)\|_{\mathbf{C}}]}\big]\cdot\mathbb{E}\big[e^{\varepsilon'\|B_0(T)\|_{\mathbf{C}}}\big].$$

Moreover, by Hölder's inequality we have,

$$\mathbb{E}\big[e^{\varepsilon'\rho^2(g_P(T))}\big] \leqslant \mathbb{E}\big[e^{\varepsilon'\|B_0(T)\|_{\mathbf{C}}}\big]\sqrt{\mathbb{E}\big[e^{2\varepsilon'\|B_P(T)\|_W^2}\big]\cdot\mathbb{E}\big[e^{\varepsilon'\|N_P(T)\|_{\mathbf{C}}}\big]}$$
$$\leqslant \mathbb{E}\big[e^{\varepsilon'\|B_0(T)\|_{\mathbf{C}}}\big]\sqrt{\mathbb{E}\big[e^{2\varepsilon'\|B(T)\|_W^2}\big]\cdot\sup_{P'\in\mathrm{Proj}(W)}\mathbb{E}\big[e^{\varepsilon'\|N_{P'}(T)\|_{\mathbf{C}}}\big]}$$

wherein we have made use of Proposition 4.14 to conclude that

$$\mathbb{E}\big[e^{2\varepsilon'\|B_P(T)\|_W^2}\big] \leqslant \mathbb{E}\big[e^{2\varepsilon'\|B(T)\|_W^2}\big] = \mathbb{E}\big[e^{2\varepsilon' T\|B(1)\|_W^2}\big]$$

which is finite by Fernique's theorem provided $2\varepsilon = 2\varepsilon' T < \delta'$ for some $\delta' > 0$. Similarly by Proposition 4.13,

$$\sup_{P'\in\mathrm{Proj}(W)}\mathbb{E}\big[e^{\varepsilon'\|N_{P'}(T)\|_{\mathbf{C}}}\big] < \infty$$

provided $\varepsilon = \varepsilon' T < \frac{\pi}{4\sqrt{\gamma}}$. The assertion in Eq. (4.45) now follows from these observations and the fact that $\mathbb{E}[e^{\varepsilon' \|B_0(T)\|_{\mathbf{C}}}] < \infty$ for all $\varepsilon' > 0$. □

5. Path space quasi-invariance

Notation 5.1. Let $W_T(G)$ denote the collection of continuous paths, $g:[0,T] \to G$ such that $g(0) = \mathbf{e}$. Moreover, if V is a separable Hilbert space, let $\mathcal{H}_T(V)$ denote the collection of absolutely continuous functions (see [14, pp. 106–107]), $h:[0,T] \to V$ such that $h(0) = 0$ and

$$\|h\|_{\mathcal{H}_T(V)} := \left(\int_0^T \|\dot{h}(t)\|_V^2 \, dt \right)^{1/2} < \infty.$$

By polarization, we endow $\mathcal{H}_T(V)$ with the inner product

$$\langle h, k \rangle_{\mathcal{H}_T(V)} = \int_0^T \langle \dot{h}(t), \dot{k}(t) \rangle_V \, dt.$$

Theorem 5.2 *(Path space quasi-invariance). Suppose $T > 0$, $k(\cdot) = (A(\cdot), a(\cdot)) \in \mathcal{H}_T(\mathfrak{g}_{CM})$ (thought of as a finite energy path in G_{CM}), and $g(\cdot)$ is the G-valued Brownian motion in Eq. (4.10). Then over the finite time interval, $[0,T]$, the laws of $k \cdot g$ and g are equivalent, i.e. they are mutually absolutely continuous relative to one another. More precisely, if $F: W_T(G) \to [0,\infty]$ is a measurable function, then*

$$\mathbb{E}\big[F(k \cdot g)\big] = \mathbb{E}\big[\tilde{Z}_k(B, B_0) F(g)\big], \tag{5.1}$$

where

$$\tilde{Z}_k(B, B_0) := \exp\left(\begin{array}{l} \int_0^T \langle \dot{A}(t), dB(t) \rangle_H - \frac{1}{2} \int_0^T \|\dot{A}(t)\|_H^2 \, dt \\ \quad + \int_0^T \langle \dot{a}(t) + \frac{1}{2}\omega(A(t) - 2B(t), \dot{A}(t)), dB_0(t) \rangle_{\mathbf{C}} \\ \quad - \frac{1}{2} \int_0^T \|\dot{a}(t) + \frac{1}{2}\omega(A(t) - 2B(t), \dot{A}(t))\|_{\mathbf{C}}^2 \, dt \end{array} \right). \tag{5.2}$$

Moreover, Eq. (5.2) is valid for all measurable functions, $F: W_T(G) \to \mathbb{C}$ such that

$$\mathbb{E}\big[\big|F(k \cdot g)\big|\big] = \mathbb{E}\big[\tilde{Z}_k(B, B_0)\big|F(g)\big|\big] < \infty.$$

Proof. The Cameron–Martin theorem states (see for example, [36, Theorem 1.2, p. 113]) that

$$\mathbb{E}\big[F(B, B_0)\big] = \mathbb{E}\big[Z_k(B, B_0) F\big((B, B_0) - k\big)\big], \tag{5.3}$$

where

$$Z_k(B, B_0) := \exp\left(\begin{array}{l} \int_0^T [\langle \dot{A}(t), dB(t) \rangle_H + \langle \dot{a}(t), dB_0(t) \rangle_{\mathbf{C}}] \\ \quad - \frac{1}{2} \int_0^T [\|\dot{A}(t)\|_H^2 + \|\dot{a}(t)\|_{\mathbf{C}}^2] \, dt \end{array} \right). \tag{5.4}$$

Since

$$(k\cdot g)(t)=\left(B(t)+A(t),B_0(t)+a(t)+\frac{1}{2}\int_0^t\omega\big(B(\tau),dB(\tau)\big)+\frac{1}{2}\omega\big(A(t),B(t)\big)\right) \tag{5.5}$$

is mapped to

$$\left(B(t),B_0(t)+\frac{1}{2}\int_0^t\omega\big((B-A)(\tau),d(B-A)(\tau)\big)+\frac{1}{2}\omega\big(A(t),(B-A)(t)\big)\right)$$

under the transformation $B\to B-A$ and $B_0\to B_0-a$, we may conclude from Eq. (5.3) that

$$\mathbb{E}\big[F(k\cdot g)\big]=\mathbb{E}\big[Z_k(B,B_0)F(B,B_0+c)\big], \tag{5.6}$$

where

$$c(t)=\frac{1}{2}\int_0^t\omega\big((B-A)(\tau),d(B-A)(\tau)\big)+\frac{1}{2}\omega\big(A(t),(B-A)(t)\big).$$

By taking the differential of c, one easily shows that

$$c(t)=\frac{1}{2}\int_0^t\omega\big(B(\tau),dB(\tau)\big)+u_B(t),$$

where

$$u_B(t):=\frac{1}{2}\int_0^t\omega\big(A(\tau)-2B(\tau),\dot{A}(\tau)\big)\,d\tau. \tag{5.7}$$

Hence Eq. (5.6) may be rewritten as

$$\mathbb{E}\big[F(k\cdot g)\big]=\mathbb{E}\left[Z_k(B,B_0)F\left(B,B_0+u_B+\frac{1}{2}\int_0^{\cdot}\omega\big(B(t),dB(t)\big)\right)\right]. \tag{5.8}$$

Freezing the integration over B (i.e. using Fubini's theorem) we may use the Cameron–Martin theorem one more time to make the transformation, $B_0\to B_0-u_B$. Doing so gives

$$\begin{aligned}\mathbb{E}\big[F(k\cdot g)\big]&=\mathbb{E}\left[\tilde{Z}_k(B,B_0)F\left(\left(B,B_0+\frac{1}{2}\int_0^{\cdot}\omega\big(B(t),dB(t)\big)\right)\right)\right]\\&=\mathbb{E}\big[\tilde{Z}_k(B,B_0)F(g)\big],\end{aligned} \tag{5.9}$$

where

$$\tilde{Z}_k(B, B_0) := Z_k(B, B_0 - u_B) \exp\left(\int_0^T \langle \dot{u}_B(t), dB_0(t) \rangle_{\mathbf{C}} - \frac{1}{2} \int_0^T \|\dot{u}_B(t)\|_{\mathbf{C}}^2 \, dt \right). \tag{5.10}$$

A little algebra shows that $\tilde{Z}_k(B, B_0)$ defined in Eq. (5.10) may be expressed as in Eq. (5.2). □

Remark 5.3. The above proof fails if we try to use it to prove the right quasi-invariance on the path space measure, i.e. that $g \cdot k$ has a law which is absolutely continuous to that of g. In this case

$$(g \cdot k)(t) = \left(B(t) + A(t), B_0(t) + a(t) + \frac{1}{2} \int_0^t \omega\big(B(\tau), dB(\tau)\big) - \frac{1}{2} \omega\big(A(t), B(t)\big) \right)$$

and then making the transformation, $B \to B - A$ and $B_0 \to B_0 - a$ gives

$$\mathbb{E}\big[F(g \cdot k)\big] = \mathbb{E}\big[Z_k(B, B_0) F(B, B_0 + c)\big],$$

where

$$c(t) = \frac{1}{2} \int_0^t \omega\big(B(\tau), dB(\tau)\big) + u_B(t)$$

and

$$u_B(t) = \frac{1}{2} \int_0^t \big[\omega(A, dA) - 2\omega(A, dB)\big].$$

The argument breaks down at this point since u_B is no longer absolutely continuous in t. Hence we can no longer use the Cameron–Martin theorem to translate away the u_B term.

Proposition 5.4. *There exists a $\delta > 0$ and a function $C(p, u) \in (0, \infty]$, for $1 < p < \infty$ and $0 \leqslant u < \infty$, which is non-decreasing in each of its variables, $C(p, u) < \infty$ whenever*

$$p \leqslant \frac{1}{2}\big(1 + \sqrt{1 + \delta/u}\big), \tag{5.11}$$

and,

$$\mathbb{E}\big[\tilde{Z}_k(B, B_0)^p\big] \leqslant C\big(p, \|k\|_{\mathcal{H}_T(\mathfrak{g}_{CM})}\big) \quad \text{for all } k \in \mathcal{H}_T(\mathfrak{g}_{CM}). \tag{5.12}$$

Proof. For the purposes of this proof, let $\mathbb{E}_{B_0}$ and $\mathbb{E}_B$ denote the expectation relative to B_0 and B, respectively, so that by Fubini's theorem $\mathbb{E} = \mathbb{E}_{B_0}\mathbb{E}_B = \mathbb{E}_B\mathbb{E}_{B_0}$, We may write $\tilde{Z}_k(B, B_0)$ as

$$\tilde{Z}_k(B, B_0) := \zeta(B)\exp\left(\int_0^T \langle \dot{a}(t) + \dot{u}_B(t), dB_0(t)\rangle_{\mathbf{C}}\right)$$

where

$$\zeta(B) := \exp\left(\int_0^T \langle \dot{A}(t), dB(t)\rangle_H - \frac{1}{2}\int_0^T \|\dot{A}(t)\|_H^2\,dt - \frac{1}{2}\int_0^T \|\dot{a}(t) + \dot{u}_B(t)\|_{\mathbf{C}}^2\,dt\right)$$

and $u_B(t)$ is as in Eq. (5.7). Hence it follows that,

$$\begin{aligned}\mathbb{E}_{B_0}\left[\tilde{Z}_k(B, B_0)^p\right] &= \zeta^p(B)\mathbb{E}_{B_0}\left[\exp\left(p\int_0^T \langle \dot{a}(t) + \dot{u}_B(t), dB_0(t)\rangle_{\mathbf{C}}\right)\right] \\ &= \zeta^p(B)\exp\left(\frac{p^2}{2}\int_0^T \|\dot{a}(t) + \dot{u}_B(t)\|_C^2\,dt\right) = UV, \qquad (5.13)\end{aligned}$$

where

$$U := \exp\left(p\left(\int_0^T \langle \dot{A}(t), dB(t)\rangle_H - \frac{1}{2}\int_0^T \|\dot{A}(t)\|_H^2\,dt\right)\right)$$

and

$$V = \exp\left(\frac{p^2 - p}{2}\int_0^T \|\dot{a}(t) + \dot{u}_B(t)\|_{\mathbf{C}}^2\,dt\right).$$

Note that when $p = 1$, Eq. (5.13) becomes

$$\mathbb{E}_{B_0}\left[\tilde{Z}_k(B, B_0)\right] = \exp\left(\int_0^T \langle \dot{A}(t), dB(t)\rangle_H - \frac{1}{2}\int_0^T \|\dot{A}(t)\|_H^2\,dt\right),$$

from which it easily follows that

$$\mathbb{E}\left[\tilde{Z}_k(B, B_0)\right] = \mathbb{E}_B\mathbb{E}_{B_0}\left[\tilde{Z}_k(B, B_0)\right] = 1.$$

Now suppose that $p > 1$. By the Cauchy–Schwarz inequality,

$$\mathbb{E}\left[\tilde{Z}_k(B, B_0)^p\right] = \mathbb{E}_B[UV] \leqslant \left(\mathbb{E}_B\left[V^2\right]\right)^{1/2}\left(\mathbb{E}_B\left[U^2\right]\right)^{1/2}.$$

Because

$$\mathbb{E}U^2 = \exp\left(-p\int_0^T \left\|\dot{A}(t)\right\|_H^2 dt\right)\mathbb{E}\left[\exp\left(2p\int_0^T \left\langle \dot{A}(t), dB(t)\right\rangle_H\right)\right]$$
$$= \exp\left(p\|A\|_{\mathcal{H}_T(H)}^2\right) \leqslant \exp\left(p\|k\|_{\mathcal{H}_T(\mathfrak{g}_{CM})}^2\right) < \infty,$$

we have reduced the problem to estimating $\mathbb{E}V^2$. By elementary estimates we have

$$\begin{aligned}\left\|\dot{u}_B(t)\right\|_{\mathbf{C}}^2 &= \frac{1}{4}\left\|\omega\left(A(t) - 2B(t), \dot{A}(t)\right)\right\|_{\mathbf{C}}^2\\ &\leqslant \frac{1}{4}\|\omega\|_0^2\left\|A(t) - 2B(t)\right\|_W^2\left\|\dot{A}(t)\right\|_W^2\\ &\leqslant \frac{1}{2}\|\omega\|_0^2\left\|\dot{A}(t)\right\|_W^2\left(\left\|A(t)\right\|_W^2 + 4\left\|B(t)\right\|_W^2\right)\end{aligned}$$

and hence

$$\begin{aligned}\left\|\dot{a}(t) + \dot{u}_B(t)\right\|^2 &\leqslant 2\left\|\dot{a}(t)\right\|_{\mathbf{C}}^2 + 2\left\|\dot{u}_B(t)\right\|_{\mathbf{C}}^2\\ &= 2\left\|\dot{a}(t)\right\|_{\mathbf{C}}^2 + \|\omega\|_0^2\left\|\dot{A}(t)\right\|_W^2\left(\left\|A(t)\right\|_W^2 + 4\sup_{0\leqslant t\leqslant T}\left\|B(t)\right\|_W^2\right).\end{aligned} \tag{5.14}$$

By Eq. (2.10) there exits $c < \infty$ such that $\|\cdot\|_W \leqslant c\|\cdot\|_H$. Since

$$\left\|A(t)\right\|_H \leqslant \int_0^T \left\|\dot{A}(\tau)\right\|_H d\tau \leqslant \sqrt{T}\|A\|_{\mathcal{H}_T(H)},$$

we find

$$V^2 \leqslant C\exp\left(4\left(p^2 - p\right)c^2\|\omega\|_0^2\|A\|_{\mathcal{H}_T(H)}^2\sup_{0\leqslant t\leqslant T}\left\|B(t)\right\|_W^2\right),$$

where

$$\begin{aligned}C &= \exp\left(\left(p^2 - p\right)\left(2\|a\|_{\mathcal{H}_T(\mathbf{C})}^2 + c^4T\|\omega\|_0^2\|A\|_{\mathcal{H}_T(H)}^4\right)\right)\\ &\leqslant C'\left(p, \|k\|_{\mathcal{H}_T(\mathfrak{g}_{CM})}\right) < \infty.\end{aligned}$$

Now by Fernique's theorem as in Eq. (2.4) there exists $\delta' > 0$ such that

$$M := \mathbb{E}\left[\exp\left(\delta'\sup_{0\leqslant t\leqslant T}\left\|B(t)\right\|_W^2\right)\right] < \infty$$

and hence it follows that

$$\mathbb{E}V^2 \leqslant C'\left(p, \|k\|_{\mathcal{H}_T(\mathfrak{g}_{CM})}\right)\cdot M < \infty$$

provided

$$4(p^2-p)c^2\|\omega\|_0^2\|A\|_{\mathcal{H}_T(H)}^2 \leqslant 4(p^2-p)c^2\|\omega\|_0^2\|k\|_{\mathcal{H}_T(\mathfrak{g}_{CM})}^2 \leqslant \delta'.$$

The latter condition holds provided

$$p \leqslant \frac{1+\sqrt{1+\delta/\|k\|_{\mathcal{H}_T(\mathfrak{g}_{CM})}^2}}{2},$$

where $\delta := (c^2\|\omega\|_0^2)^{-1}\delta' > 0$. □

Definition 5.5. We will say that a function, $F : W_T(G) \to \mathbb{R}$ ($W_T(G)$ as in Notation 5.1) is *polynomially bounded* if there exist constants $K, M < \infty$ such that

$$\left|F(g)\right| \leqslant K\Bigl(1+\sup_{t\in[0,T]}\left\|g(t)\right\|_{\mathfrak{g}}\Bigr)^M \quad \text{for all } g \in W_T(G). \tag{5.15}$$

Given a finite energy path, $k(t) = (A(t), a(t)) \in \mathfrak{g}_{CM}$, we say that F is right k differentiable if

$$\left.\frac{d}{ds}\right|_0 F\bigl((sk)\cdot g\bigr) =: (\hat{k}F)(g)$$

exists for all $g \in W_T(G)$.

Corollary 5.6 *(Path space integration by parts). Let $k(\cdot) = (A(\cdot), a(\cdot)) \in \mathcal{H}_T(\mathfrak{g}_{CM})$ and $F : W_T(G) \to \mathbb{R}$ be a k-differentiable function such that F and $\hat{k}F$ are polynomial bounded functions on $W_T(G)$. Then*

$$\mathbb{E}\bigl[(\hat{k}F)(g)\bigr] = \mathbb{E}\bigl[F(g)z_k\bigr], \tag{5.16}$$

where

$$z_k := \int_0^T \bigl[\bigl\langle \dot{A}(t), dB(t)\bigr\rangle_H + \bigl\langle \dot{a}(t) - \omega\bigl(B(t), \dot{A}(t)\bigr), dB_0(t)\bigr\rangle_{\mathbf{C}}\bigr]. \tag{5.17}$$

Moreover, $\mathbb{E}|z_k|^p < \infty$ for all $p \in [1,\infty)$.

Proof. From Theorem 5.2, we have that for any $s \in \mathbb{R}$

$$\mathbb{E}\bigl[F\bigl((sk)\cdot g\bigr)\bigr] = \mathbb{E}\bigl[\tilde{Z}_{sk}(B, B_0)F(g)\bigr]. \tag{5.18}$$

Formally differentiating this identity at $s = 0$ and interchanging the derivatives with the expectations immediately leads to Eq. (5.16). To make this rigorous we need only to verify that derivative interchanges are permissible. From Eqs. (3.8) and (5.15), there exists $C(k) < \infty$ such that

$$\begin{aligned}\sup_{|s|\leqslant 1}\left|\frac{d}{ds}F\big((sk)\cdot g\big)\right| &= \sup_{|s|\leqslant 1}\big|(\hat{k}F)\big((sk)\cdot g\big)\big| \\ &\leqslant K\sup_{|s|\leqslant 1}\Big(1+\sup_{t\in[0,T]}\big\|\big[sk(t)\big]\cdot g(t)\big\|_{\mathfrak{g}}\Big)^M \\ &\leqslant C(k)\Big(1+\sup_{t\in[0,T]}\big\|g(t)\big\|_{\mathfrak{g}}\Big)^M,\end{aligned}$$

wherein the last expression is integrable by Fernique's theorem and the moment estimate in Proposition 4.1. Therefore,

$$\frac{d}{ds}\bigg|_0 \mathbb{E}\big[F\big((sk)\cdot g\big)\big] = \mathbb{E}\bigg[\frac{d}{ds}\bigg|_0 F\big((sk)\cdot g\big)\bigg] = \mathbb{E}\big[(\hat{k}F)(g)\big].$$

To see that we may also differentiate the right-hand side of Eq. (5.18), observe that

$$\tilde{Z}_{sk}(B,B_0) = \exp\big(sz_k + s^2\beta + s^3\gamma + s^4\kappa\big),$$

where

$$\begin{aligned}\beta &= -\frac{1}{2}\int_0^T \big\|\dot{A}(t)\big\|_H^2\,dt + \frac{1}{2}\int_0^T \big\langle\omega\big(A(t),\dot{A}(t)\big),dB_0(t)\big\rangle_{\mathbf{C}} \\ &\quad - \frac{1}{2}\int_0^T \big\|\dot{a}(t)-\omega\big(B(t),\dot{A}(t)\big)\big\|_{\mathbf{C}}^2\,dt, \\ \gamma &= -\frac{1}{2}\int_0^T \operatorname{Re}\big\langle\dot{a}(t)-\omega\big(B(t),\dot{A}(t)\big),\omega\big(A(t),\dot{A}(t)\big)\big\rangle_{\mathbf{C}}\,dt,\end{aligned}$$

and

$$\kappa = -\frac{1}{8}\int_0^T \big\|\omega\big(A(t),\dot{A}(t)\big)\big\|_{\mathbf{C}}^2\,dt.$$

Using Fernique's theorem again and estimates similar to those used in the proof of Proposition 5.4, one shows for any $p\in[1,\infty)$ that there exists $s_0(p)>0$ such that

$$\mathbb{E}\bigg[\sup_{|s|\leqslant s_0(p)}\bigg|\frac{d}{ds}\tilde{Z}_{sk}(B,B_0)\bigg|^p\bigg] < \infty.$$

Therefore we may differentiate past the expectation to find

$$\frac{d}{ds}\bigg|_0 \mathbb{E}\big[F(g)\tilde{Z}_{sk}(B,B_0)\big] = \mathbb{E}\bigg[F(g)\frac{d}{ds}\bigg|_0 \tilde{Z}_{sk}(B,B_0)\bigg] = \mathbb{E}\big[F(g)z_k\big].$$

The fact that z_k has finite moments of all orders follows by the martingale arguments along with Nelson's theorem as described in the proof of Lemma 4.7. Alternatively, observe that $\int_0^T \langle \dot{A}(t), dB(t)\rangle_H$ is Gaussian and hence has finite moments of all orders. If we let $M_t := \int_0^t \langle \dot{a} - \omega(B, \dot{A}), dB_0\rangle_{\mathbf{C}}$, then M is a martingale such that

$$\langle M\rangle_T = \int_0^T \left\| \dot{a}(t) - \omega\big(B(t), \dot{A}(t)\big)\right\|_C^2 dt \leqslant C\Big(1 + \max_{0\leqslant t\leqslant T}\left\|B(t)\right\|_W^2\Big).$$

So by Fernique's theorem, $\mathbb{E}[\langle M\rangle_T^p] < \infty$ for all $p < \infty$ and hence by the Burkholder–Davis–Gundy inequalities, $\mathbb{E}|M_T|^p < \infty$ for all $1 \leqslant p < \infty$. □

6. Heat kernel quasi-invariance

In this section we will use the results of Section 5 to prove both quasi-invariance of the heat kernel measures, $\{\nu_T\}_{T>0}$, relative to left and right translations by elements of G_{CM}.

Theorem 6.1 *(Left quasi-invariance of the heat kernel measure). Let $T > 0$ and $(A, a) \in G_{\mathrm{CM}}$. Then $(A, a)\cdot g(T)$ and $g(T)$ have equivalent laws. More precisely, if $f : G \to [0, \infty]$ is a measurable function, then*

$$\mathbb{E}\big[f\big((A, a)\cdot g(T)\big)\big] = \mathbb{E}\big[f\big(g(T)\big)\bar{Z}_k\big(g(T)\big)\big], \tag{6.1}$$

where

$$\bar{Z}_k\big(g(T)\big) = \mathbb{E}\big[\zeta_{(A,a)}(B, B_0)\big|\sigma\big(g(T)\big)\big] \tag{6.2}$$

and

$$\ln \zeta_{(A,a)}(B, B_0) := \frac{1}{T}\big\langle A, B(T)\big\rangle_H - \frac{\|A\|_H^2}{2T^2} + \frac{1}{T}\int_0^T \big\langle a - \omega\big(B(t), A\big), dB_0(t)\big\rangle_{\mathbf{C}} - \frac{1}{2T^2}\int_0^T \big\|a - \omega\big(B(t), A\big)\big\|_{\mathbf{C}}^2 dt. \tag{6.3}$$

Proof. An application of Theorem 5.2 with $F(g) := f(g(T))$ and $k(t) := \frac{t}{T}(A, a)$ implies

$$\begin{aligned}\mathbb{E}\big[f\big((A, a)\cdot g(T)\big)\big] &= \mathbb{E}\big[F(\mathbf{k}\cdot g)\big] = \mathbb{E}\big[\tilde{Z}_k(B, B_0)\cdot F(g)\big]\\ &= \mathbb{E}\big[\tilde{Z}_k(B, B_0) f\big(g(T)\big)\big],\end{aligned} \tag{6.4}$$

where after a little manipulation one shows, $\tilde{Z}_k(B, B_0) = \zeta_{(A,a)}(B, B_0)$. By conditioning on $\sigma(g(T))$ we can also write Eq. (6.4) as in Eq. (6.1). □

Corollary 6.2 *(Right quasi-invariance of the heat kernel measure). The heat kernel measure, ν_T, is also quasi-invariant under right translations, and*

$$\frac{d\nu_T \circ r_k^{-1}}{d\nu_T}(g) = \bar{Z}_{k^{-1}}\big(g^{-1}\big), \tag{6.5}$$

where

$$\bar{Z}_k = d\nu_T \circ l_k^{-1}/d\nu_T$$

is as in Theorem 6.1.

Proof. Recall from Corollary 4.9 that ν_T is invariant under the inversion map, $g \to g^{-1}$. From this observation and Theorem 6.1 it follows that ν_T is also quasi-invariant under right translations of elements of G_{CM}. In more detail, if $k \in G_{\mathrm{CM}}$ and $f : G \to \mathbb{R}$ is a bounded measurable function, then

$$\begin{aligned}\int_G f(g \cdot k)\, d\nu_T(g) &= \int_G f\big(g^{-1} \cdot k\big)\, d\nu_T(g) = \int_G f\big(\big(k^{-1} g\big)^{-1}\big)\, d\nu_T(g) \\ &= \int_G f\big(g^{-1}\big)\bar{Z}_{k^{-1}}(g)\, d\nu_T(g) = \int_G f(g)\bar{Z}_{k^{-1}}\big(g^{-1}\big)\, d\nu_T(g).\end{aligned}$$

Eqs. (6.5) is a consequence of this identity. □

Just like in the case of abstract Wiener spaces we have the following strong converses of Theorem 6.1 and Corollary 6.2.

Proposition 6.3. *Suppose that $k \in G \setminus G_{\mathrm{CM}}$ and $T > 0$, then $\nu_T \circ l_k^{-1}$ and ν_T are singular and $\nu_T \circ r_k^{-1}$ and ν_T are singular.*

Proof. Let $k = (A, a) \in G \setminus G_{\mathrm{CM}}$ with $a \in \mathbf{C}$ and $A \in W \setminus H$. Given a measurable subset, $V \subset W$, we have

$$\nu_T(V \times \mathbf{C}) = P\big(B(T) \in V\big) =: \mu_T(V),$$

where μ_T is Wiener measure on W with variance T. It is well known (see e.g. Corollary 2.5.3 in [8]) that if $A \in W \setminus H$ that $\mu_T(\cdot - A)$ is singular relative to $\mu_T(\cdot)$, i.e. we may partition W into two disjoint measurable sets, W_0 and W_1 such that $\mu_T(W_0) = 1 = \mu_T(W_1 - A)$. A simple computation shows for any $V \subset W$ that

$$l_k^{-1}(V \times \mathbf{C}) = r_k^{-1}(V \times \mathbf{C}) = (V - A) \times \mathbf{C}.$$

Thus if we define $G_i := W_i \times \mathbf{C}$ for $i = 0, 1$, we have that G is the disjoint union of G_0 and G_1 and $\nu_T(G_0) = \mu_T(W_0) = 1$ while

$$\nu_T\big(r_k^{-1}(G_1)\big) = \nu_T\big(l_k^{-1}(G_1)\big) = \nu_T\big((W_1 - A) \times \mathbf{C}\big) = \mu_T(W_1 - A) = 1. \qquad \square$$

Corollary 6.4 *(Right heat kernel integration by parts). Let $k := (A, a) \in \mathfrak{g}_{CM}$ and suppose that $f : G \to \mathbb{C}$ is a smooth function such that f and $\hat{k} f$ are polynomially bounded. Then*

$$\mathbb{E}\big[(\hat{k} f)\big(g(T)\big)\big] = \mathbb{E}\big[f\big(g(T)\big) z_k\big],$$

where $\hat{k} f(g) := \frac{d}{ds}|_0 f((sk)g)$ and

$$z_k := \frac{1}{T}\left[\big\langle A, B(T)\big\rangle_H + \big\langle a, B_0(T)\big\rangle_{\mathbf{C}} - \int_0^T \big\langle \omega\big(B(t), A\big), dB_0(t)\big\rangle_{\mathbf{C}}\right].$$

Moreover, with $\nu_T := \mathrm{Law}(g(T))$, the above formula gives,

$$\int_G (\hat{k} f)\, d\nu_T(g) = \int_G f(g) \bar{z}_k(g)\, d\nu_T(g),$$

where

$$\bar{z}_k\big(g(T)\big) := \mathbb{E}\big(z_k | \sigma\big(g(T)\big)\big). \tag{6.6}$$

Proof. This is a special case of Corollary 5.6, with $k(t) := \frac{t}{T}(A, a)$ and $F(g) := f(g(T))$. $\square$

Corollary 6.5 *(Left heat kernel integration by parts). Let $k := (A, a) \in \mathfrak{g}_{CM}$ and suppose that $f : G \to \mathbb{C}$ is a smooth function such that f and $\tilde{k} f$ are polynomially bounded. Then*

$$\int_G (\tilde{k} f)\, d\nu_T(g) = \int_G f(g) \bar{z}_k^l(g)\, d\nu_T(g),$$

where $\tilde{k} f(g) := \frac{d}{ds}|_0 f(g(sk))$ and

$$\bar{z}_k^l(g) = -\bar{z}_k\big(g^{-1}\big), \tag{6.7}$$

where $\bar{z}_k$ is defined in Eq. (6.6).

Proof. Let $u(g) := f(g^{-1})$ so that $f(g) = u(g^{-1})$. Then

$$(\tilde{k} f)(g) = \frac{d}{ds}\bigg|_0 f\big(g \cdot (sk)\big) = \frac{d}{ds}\bigg|_0 u\big((-sk) \cdot g^{-1}\big) = -(\hat{k} u)\big(g^{-1}\big).$$

Therefore by Corollary 6.4 and two uses of Corollary 4.9 we find

$$\begin{aligned}\int_G (\tilde{k} f)\, d\nu_T(g) &= -\int_G (\hat{k} u)\big(g^{-1}\big)\, d\nu_T(g) = -\int_G (\hat{k} u)(g)\, d\nu_T(g) \\ &= -\int_G u(g) \bar{z}_k(g)\, d\nu_T(g) = -\int_G f\big(g^{-1}\big)\, \bar{z}_k(g)\, d\nu_T(g) \\ &= -\int_G f(g) \bar{z}_k\big(g^{-1}\big)\, d\nu_T(g). \qquad \square\end{aligned}$$

Definition 6.6. A *cylinder polynomial* is a cylinder function, $f = F \circ \pi_P : G \to \mathbb{C}$, where $P \in \operatorname{Proj}(W)$ and F is a real or complex polynomial function on $PH \times \mathbf{C}$.

Corollary 6.7 *(Closability of the Dirichlet form). Given real-valued cylindrical polynomials, u, v on G, let*

$$\mathcal{E}_T^0(u, v) := \int_G \langle \operatorname{grad} u, \operatorname{grad} v\rangle_H \, d\nu_T,$$

where $\operatorname{grad} u : G \to \mathfrak{g}_{CM}$ *is the gradient of u defined by*

$$\langle \operatorname{grad} u, k\rangle_{\mathfrak{g}_{CM}} = \tilde{k}u \quad \text{for all } k \in \mathfrak{g}_{CM}.$$

Then $\mathcal{E}_T^0$ is closable and its closure, $\mathcal{E}_T$, is a Dirichlet form on $\operatorname{Re} L^2(G, \nu_T)$.

Proof. The closability of $\mathcal{E}_T^0$ is equivalent to the closability of the gradient operator,

$$\operatorname{grad} : L^2(\nu_T) \to L^2(\nu_T) \otimes \mathfrak{g}_{CM},$$

with the domain, $\mathcal{D}(\operatorname{grad})$, being the space of cylinder polynomials on G. To check the latter statement it suffices to show that grad has a densely defined adjoint which is easily accomplished. Indeed, if $k \in \mathfrak{g}_{CM}$ and u and v are cylinder polynomials, then

$$\begin{aligned}
\langle \operatorname{grad} u, v \cdot k\rangle_{L^2(\nu_T)\otimes \mathfrak{g}_{CM}} &= \int_G \tilde{k}u \cdot v \, d\nu_T \\
&= \int_G \big[\tilde{k}(u \cdot v) - u \cdot \tilde{k}v\big] \, d\nu_T \\
&= \big\langle u, -\tilde{k}v + \bar{z}_k^l v\big\rangle_{L^2(\nu_T)},
\end{aligned}$$

wherein we have used the product rule in the second equality and Corollary 6.5 for the third. This shows that $v \cdot k$ is contained in the domain of grad^* and $\operatorname{grad}^*(v \cdot k) = -\tilde{k}v + \bar{z}_k^l v$, where z_k^l is as in Eq. (6.7). This completes the proof since linear combination of functions of the form $v \cdot k$ with $k \in \mathfrak{g}_{CM}$ and v being a cylinder polynomial is dense in $L^2(\nu_T) \otimes \mathfrak{g}_{CM}$. □

7. The Ricci curvature on Heisenberg type groups

In this section we compute the Ricci curvature for $G(\omega)$ and its finite-dimensional approximations. This information will be used in Section 8 to prove a logarithmic Sobolev inequality for ν_T and to get detailed L^p-bounds on the Radon–Nikodym derivatives of ν_T under translations by elements from G_{CM}.

Notation 7.1. Let (W, H, ω) be as in Notation 3.1, $P \in \operatorname{Proj}(W)$, and $G_P = PW \times \mathbf{C} \subset G_{CM}$ as in Notation 3.24. We equip G_P with the left-invariant Riemannian metric induced from restriction of the (real part of the) inner product on $\mathfrak{g}_{CM} = H \times \mathbf{C}$ to $\operatorname{Lie}(G_P) = PH \times \mathbf{C}$. Further, let Ric^P denote the associated Ricci tensor at the identity in G_P.

Proposition 7.2. *If* (W, H, ω, P) *as in Notation* 7.1, $P \in \operatorname{Proj}(W)$ *is as in Eq.* (3.42), *and* $(A, a) \in PH \times \mathbf{C}$, *then*

$$\left\langle \operatorname{Ric}^P(A,a),(A,a)\right\rangle_{H\times\mathbf{C}} = \frac{1}{4}\sum_{j,k=1}^{n}\left|\left\langle \omega(e_k,e_j),a\right\rangle_{\mathbf{C}}\right|^2 - \frac{1}{2}\sum_{k=1}^{n}\left\|\omega(A,e_k)\right\|_{\mathbf{C}}^2 \tag{7.1}$$

$$= \frac{1}{4}\left\|\left\langle \omega(\cdot,\cdot),a\right\rangle_{\mathbf{C}}\right\|^2_{(PH)^*\otimes(PH)^*} - \frac{1}{2}\left\|\omega(A,\cdot)\right\|^2_{(PH)^*\otimes\mathbf{C}}. \tag{7.2}$$

Proof. We are going to compute Ric^P using the formula in Eq. (B.3) of Appendix B. If $\{f_\ell\}_{\ell=1}^{\dim\mathbf{C}}$ is an orthonormal basis for $\mathbf{C}$, then

$$\sum_{k=1}^{n}\left\|\operatorname{ad}_{(e_k,0)}(A,a)\right\|^2_{H\times\mathbf{C}} + \sum_{\ell=1}^{\dim\mathbf{C}}\left\|\operatorname{ad}_{(0,f_\ell)}(A,a)\right\|^2_{H\times\mathbf{C}} = \sum_{k=1}^{n}\left\|\omega(e_k,A)\right\|^2_{\mathbf{C}}. \tag{7.3}$$

If $(B,b) \in PH \times \mathbf{C}$, then

$$\begin{aligned}
\operatorname{ad}^*_{(B,b)}(A,a) &= \sum_{j=1}^{n}\left\langle \operatorname{ad}^*_{(B,b)}(A,a),(e_j,0)\right\rangle_{\mathfrak{g}_{CM}}(e_j,0) + \sum_{\ell=1}^{\dim\mathbf{C}}\left\langle \operatorname{ad}^*_{(B,b)}(A,a),(0,f_\ell)\right\rangle_{\mathfrak{g}_{CM}}(0,f_\ell) \\
&= \sum_{j=1}^{n}\left\langle (A,a),\left[(B,b),(e_j,0)\right]\right\rangle_{\mathfrak{g}_{CM}}(e_j,0) \\
&\quad + \sum_{\ell=1}^{\dim\mathbf{C}}\left\langle (A,a),\left[(B,b),(0,f_\ell)\right]\right\rangle_{\mathfrak{g}_{CM}}(0,f_\ell) \\
&= \sum_{j=1}^{n}\left\langle (A,a),\left(0,\omega(B,e_j)\right)\right\rangle_{\mathfrak{g}_{CM}}(e_j,0) = \sum_{j=1}^{n}\left(a,\omega(B,e_j)\right)_{\mathbf{C}}(e_j,0).
\end{aligned}$$

This then immediately implies

$$\sum_{k=1}^{n}\left\|\operatorname{ad}^*_{(e_k,0)}(A,a)\right\|^2_{\mathfrak{g}_{CM}} + \sum_{\ell=1}^{\dim\mathbf{C}}\left\|\operatorname{ad}^*_{(0,f_\ell)}(A,a)\right\|^2_{\mathfrak{g}_{CM}} = \sum_{k=1}^{n}\sum_{j=1}^{n}\left\langle a,\omega(e_k,e_j)\right\rangle^2_{\mathbf{C}}. \tag{7.4}$$

Using Eqs. (7.3) and (7.4) with the formula for the Ricci tensor in Eq. (B.3) of Appendix B implies Eq. (7.1). □

Corollary 7.3. *For* $P \in \operatorname{Proj}(W)$ *as in* (3.42), *let*

$$k_P(\omega) := -\frac{1}{2}\sup\left\{\left\|\omega(\cdot,A)\right\|^2_{(PH)^*\otimes\mathbf{C}}\colon A \in PH,\ \|A\|_{PH} = 1\right\}. \tag{7.5}$$

Also let

$$k(\omega) := -\frac{1}{2}\sup_{\|A\|_H=1}\left\|\omega(\cdot,A)\right\|^2_{H^*\otimes\mathbf{C}} \geqslant -\frac{1}{2}\|\omega\|_2^2 > -\infty. \tag{7.6}$$

Then $k_P(\omega)$ is the largest constant $k \in \mathbb{R}$ such that

$$\left\langle \mathrm{Ric}^P(A,a),(A,a)\right\rangle_{PH\times\mathbf{C}} \geqslant k\left\|(A,a)\right\|^2_{PH\times\mathbf{C}} \quad \textit{for all } (A,a)\in PH\times\mathbf{C} \tag{7.7}$$

and $k(\omega)$ is the largest constant $k \in \mathbb{R}$ such that Eq. (7.7) *holds uniformly for all $P \in \mathrm{Proj}(W)$.*

Proof. Let us observe that by Eq. (7.1)

$$\frac{\langle \mathrm{Ric}^P(A,a),(A,a)\rangle_{PH\times\mathbf{C}}}{\|(A,a)\|^2_{PH\times\mathbf{C}}} \geqslant \frac{\langle \mathrm{Ric}^P(A,0),(A,0)\rangle_{PH\times\mathbf{C}}}{\|(A,0)\|^2_{PH\times\mathbf{C}}}$$

the optimal lower bound, $k_P(\omega)$, for Ric^p is determined by

$$\begin{aligned} k_P(\omega) &= \inf_{A\in PH\setminus\{0\}} \frac{\langle \mathrm{Ric}^P(A,0),(A,0)\rangle_{PH\times\mathbf{C}}}{\|(A,0)\|^2_{PH\times\mathbf{C}}} \\ &= \inf_{A\in PH\setminus\{0\}} \left(-\frac{1}{2}\frac{\|\omega(\cdot,A)\|^2_{(PH)^*\otimes\mathbf{C}}}{\|A\|^2_{PH}}\right) \end{aligned}$$

which is equivalent to Eq. (7.5). It is now a simple matter to check that $k(\omega) = \inf_{P\in\mathrm{Proj}(W)} k_P(\omega)$ which is the content of the last assertion of the theorem. □

In revisiting the examples from Section 3.3 we will have a number of cases where H and $\mathbf{C}$ are complex Hilbert spaces and $\omega: H\times H \to \mathbf{C}$ will be a complex bilinear form. In these cases it will be convenient to express the Ricci curvature in terms of these complex structures.

Proposition 7.4. *Suppose that H and $\mathbf{C}$ are complex Hilbert spaces, $\omega: H\times H\to\mathbf{C}$ is complex bi-linear, and $P: H\to H$ is a finite rank* (*complex linear*) *orthogonal projection. We make $G_P = PH\times\mathbf{C}$ into a Lie group using the group law in Eq.* (3.6)*. Let us endow G_P with the left-invariant Riemannian metric which agrees with $\langle\cdot,\cdot\rangle_{[\mathfrak{g}_P]_{\mathrm{Re}}} := \mathrm{Re}\langle\cdot,\cdot\rangle_{\mathfrak{g}_P}$ on $\mathfrak{g}_P = PH\times\mathbf{C}$ at the identity in G_P. Then for all $(A,a)\in\mathfrak{g}_P$,*

$$\begin{aligned} \left\langle \mathrm{Ric}^P(A,a),(A,a)\right\rangle_{[\mathfrak{g}_P]_{\mathrm{Re}}} &= \frac{1}{2}\left\|\left\langle\omega(\cdot,\cdot),a\right\rangle_{\mathbf{C}}\right\|^2_{(PH)^*\otimes(PH)^*} - \left\|\omega(A,\cdot)\right\|^2_{(PH)^*\otimes\mathbf{C}} && (7.8)\\ &= \frac{1}{2}\sum_{j,k=1}^n \left|\left\langle\omega(e_k,e_j),a\right\rangle_{\mathbf{C}}\right|^2 - \sum_{k=1}^n \left\|\omega(A,e_k)\right\|^2_{\mathbf{C}}, && (7.9) \end{aligned}$$

where $\{e_j\}_{j=1}^n$ is any orthonormal basis for PH.

Proof. Applying Eq. (7.2) with PH, $\mathbf{C}$, and $\mathfrak{g}_P$ being replaced by $(PH)_{\mathrm{Re}}$, $\mathbf{C}_{\mathrm{Re}}$, and $[\mathfrak{g}_P]_{\mathrm{Re}}$ implies

$$\left\langle \mathrm{Ric}^P(A,a),(A,a)\right\rangle_{[\mathfrak{g}_P]_{\mathrm{Re}}} = \frac{1}{4}\left\|\left\langle\omega(\cdot,\cdot),a\right\rangle_{\mathbf{C}_{\mathrm{Re}}}\right\|^2_{(PH)^*_{\mathrm{Re}}\otimes(PH)^*_{\mathrm{Re}}} - \frac{1}{2}\left\|\omega(A,\cdot)\right\|^2_{(PH)^*_{\mathrm{Re}}\otimes\mathbf{C}_{\mathrm{Re}}}.$$

However, by Lemma 3.17 we also know that

$$\left\|\langle\omega(\cdot,\cdot),a\rangle_{\mathbf{C}_{\mathrm{Re}}}\right\|^2_{(PH)^*_{\mathrm{Re}}\otimes(PH)^*_{\mathrm{Re}}}=2\left\|\langle\omega(\cdot,\cdot),a\rangle_{\mathbf{C}}\right\|^2_{(PH)^*\otimes(PH)^*}$$

and

$$\left\|\omega(A,\cdot)\right\|^2_{(PH)^*_{\mathrm{Re}}\otimes\mathbf{C}_{\mathrm{Re}}}=2\left\|\omega(A,\cdot)\right\|^2_{(PH)^*\otimes\mathbf{C}}$$

which completes the proof of Eq. (7.8). □

Remark 7.5. By letting $n\to\infty$ in Propositions 7.2 and 7.4, it is reasonable to interpret the Ricci tensor on G_{CM} to be determined by

$$\langle\mathrm{Ric}(A,a),(A,a)\rangle_{[\mathfrak{g}_{\mathrm{CM}}]_{\mathrm{Re}}}=\alpha_{\mathbb{F}}\left(\frac{1}{4}\left\|\langle a,\omega(\cdot,\cdot)\rangle_{\mathbf{C}}\right\|^2_{H^*\otimes H^*}-\frac{1}{2}\left\|\omega(\cdot,A)\right\|^2_{H^*\otimes\mathbf{C}}\right) \tag{7.10}$$

$$=\alpha_{\mathbb{F}}\left(\frac{1}{4}\sum_{j,k=1}^{\infty}\left|\langle a,\omega(e_k,e_j)\rangle_{\mathbf{C}}\right|^2-\frac{1}{2}\sum_{k=1}^{\infty}\left\|\omega(e_k,A)\right\|^2_{\mathbf{C}}\right), \tag{7.11}$$

where $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis for H, $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$ and $\alpha_{\mathbb{F}}$ is one or two, respectively. Moreover if $\mathbf{C}=\mathbb{F}$, then Eq. (7.10) may be written as

$$\langle\mathrm{Ric}(A,a),(A,a)\rangle_{[\mathfrak{g}_{\mathrm{CM}}]_{\mathrm{Re}}}=\alpha_{\mathbb{F}}\left(\frac{1}{4}\left\|\omega(\cdot,\cdot)\right\|^2_{H^*\otimes H^*}\cdot|a|^2-\frac{1}{2}\left\|\omega(\cdot,A)\right\|^2_{H^*}\right). \tag{7.12}$$

7.1. Examples revisited

Using Eq. (7.10), it is straightforward to compute the Ricci tensor on G for each of Examples 3.18–3.23.

Lemma 7.6. *The Ricci tensor for G_{CM} associated to each of the structures introduced in Examples* 3.18 *and* 3.19 *are given (respectively) by*

$$\langle\mathrm{Ric}(z,c),(z,c)\rangle_{\mathfrak{h}^n_{\mathbb{R}}}=\frac{nc^2}{2}-\frac{1}{2}\|z\|^2_{\mathbb{C}^n}\quad\text{for all }(z,c)\in\mathbb{C}^n\times\mathbb{R}, \tag{7.13}$$

and

$$\langle\mathrm{Ric}(z,c),(z,c)\rangle_{[\mathfrak{h}^n_{\mathbb{C}}]_{\mathrm{Re}}}=n|c|^2-\|z\|^2_{\mathbb{C}^{2n}}\quad\text{for all }(z,c)\in\mathbb{C}^{2n}\times\mathbb{C}. \tag{7.14}$$

Proof. We omit the proof of this lemma as it can be deduced from the next proposition by taking $Q=I$. □

Proposition 7.7. *The Ricci tensor for G_{CM} associated to each of the structures introduced in Examples* 3.20 *and* 3.21 *are given (respectively) by*

$$\langle\mathrm{Ric}(h,c),(h,c)\rangle_{\mathfrak{g}_{\mathrm{CM}}}=\frac{1}{2}\left[c^2\operatorname{tr}Q^2-\|Qh\|^2_H\right]\quad\text{for all }(h,c)\in H\times\mathbb{R}, \tag{7.15}$$

and

$$\left\langle \mathrm{Ric}(k_1,k_2,c),(k_1,k_2,c)\right\rangle_{[\mathfrak{g}_{CM}]_{\mathrm{Re}}} = |c|^2 \operatorname{tr} Q^2 - \|Qk_1\|_K^2 - \|Qk_2\|_K^2 \tag{7.16}$$

for all $(k_1,k_2,c)\in K\times K\times\mathbb{R}$.

Proof. We start with the proof of Eq. (7.15). In this case,

$$\begin{aligned}
\|\omega(\cdot,h)\|_{H_{\mathrm{Re}}^*}^2 &= \sum_{j=1}^\infty \left[\omega(e_j,A)^2+\omega(ie_j,A)^2\right] \\
&= \sum_{j=1}^\infty \left[\left(\mathrm{Im}\langle h,e_j\rangle_Q\right)^2+\left(\mathrm{Im}\langle h,ie_j\rangle_Q\right)^2\right] \\
&= \sum_{j=1}^\infty \left[\left(\mathrm{Im}\langle h,e_j\rangle_Q\right)^2+\left(\mathrm{Re}\langle h,e_j\rangle_Q\right)^2\right] \\
&= \sum_{j=1}^\infty \left|\langle h,e_j\rangle_Q\right|^2 = \|Qh\|_H^2
\end{aligned}$$

and from Eq. (3.36), $\|\omega\|_2^2 = 2\operatorname{tr}(Q^2)$. Using these results in Eq. (7.12) with $\mathbb{F}=\mathbb{R}$ gives Eq. (7.15) with $\mathbb{F}=\mathbb{C}$ and $H=K\times K$. Eq. (7.16) follows from Eq. (7.12) with $\mathbb{F}=\mathbb{C}$, Eq. (3.36), and the following identity:

$$\begin{aligned}
\left\|\omega\left((k_1,k_2),\cdot\right)\right\|_{H^*}^2 &= \sum_{j=1}^\infty \left(\left|\omega\left((k_1,k_2),(e_j,0)\right)\right|^2+\left|\omega\left((k_1,k_2),(0,e_j)\right)\right|^2\right) \\
&= \sum_{j=1}^\infty \left|\langle k_2,Q\bar{e}_j\rangle\right|^2 + \sum_{j=1}^\infty \left|\langle k_1,Q\bar{e}_j\rangle\right|^2 \\
&= \|Qk_1\|_K^2+\|Qk_2\|_K^2. \qquad \square
\end{aligned} \tag{7.17}$$

Proposition 7.8. *The Ricci tensor for* G_{CM} *associated to the structure introduced in Example* 3.22 *is given by*

$$\left\langle \mathrm{Ric}(v,c),(v,c)\right\rangle_{[\mathfrak{g}_{CM}]_{\mathrm{Re}}} = \sum_{j=1}^\infty q_j^2 \left\langle \mathrm{Ric}^\alpha(v_j,c),(v_j,c)\right\rangle_{V_{\mathrm{Re}}\times\mathbf{C}_{\mathrm{Re}}} \quad \forall (v,c)\in H\times\mathbb{F}, \tag{7.18}$$

where Ric^α *denotes the Ricci tensor on* $G(\alpha):=V\times\mathbf{C}$ *as is defined by Eq.* (7.19) *below.*

Proof. Using Eq. (3.38) along with the identity,

$$\begin{aligned}
\|\omega(\cdot,v)\|_{H^*\otimes\mathbf{C}}^2 &= \sum_{j=1}^\infty\sum_{a=1}^d \left\|\omega\left(u_a(j),v\right)\right\|_{\mathbf{C}}^2 = \sum_{j=1}^\infty\sum_{a=1}^d q_j^2\left\|\alpha(u_a,v_j)\right\|_{\mathbf{C}}^2 \\
&= \sum_{j=1}^\infty q_j^2 \left\|\alpha(\cdot,v_j)\right\|_{V^*\otimes\mathbf{C}}^2,
\end{aligned}$$

in Eq. (7.10) shows

$$\langle \mathrm{Ric}(v,c),(v,c)\rangle_{[\mathfrak{g}_{CM}]_{\mathrm{Re}}} = \alpha_{\mathbb{F}} \sum_{j=1}^{\infty} q_j^2 \left(\frac{1}{4} \big\| \langle \alpha(\cdot,\cdot), c\rangle_{\mathbf{C}} \big\|_2^2 - \frac{1}{2} \big\| \alpha(\cdot, v_j)\big\|_{V^*\otimes \mathbf{C}}^2 \right).$$

Moreover, by a completely analogous finite-dimensional application of Eq. (7.10), we have

$$\langle \mathrm{Ric}^{\alpha}(v_j,c),(v_j,c)\rangle_{V_{\mathrm{Re}}\times \mathbf{C}_{\mathrm{Re}}} = \alpha_{\mathbb{F}} \left(\frac{1}{4} \big\| \langle \alpha(\cdot,\cdot), c\rangle_{\mathbf{C}} \big\|_2^2 - \frac{1}{2} \big\| \alpha(\cdot, v_j)\big\|_{V^*\otimes \mathbf{C}}^2 \right). \tag{7.19}$$

Combining these two identities completes the proof. □

Proposition 7.9. *Let $\alpha : V \times V \to \mathbf{C}$ be as in Example* 3.23. *For $v \in V$, let $\alpha_v : V \to \mathbf{C}$ be defined by $\alpha_v w = \alpha(v,w)$ and $\alpha_v^* : \mathbf{C} \to V$ be its adjoint. The Ricci tensor for G_{CM} associated to the structure introduced in Example* 3.23 *is then given by*

$$\begin{aligned} \langle \mathrm{Ric}(h,c),(h,c)\rangle_{[\mathfrak{g}_{CM}]_{\mathrm{Re}}} &= \frac{1}{2}\left[\int_{[0,1]^2} (s\wedge t)^2 \, d\eta(s)\, d\bar{\eta}(t) \right] \cdot \big\| \langle c, \alpha(\cdot,\cdot)\rangle_{\mathbf{C}} \big\|_{V^*\otimes V^*}^2 \\ &\quad - \int_{[0,1]^2} (s\wedge t) \operatorname{tr}\big(\alpha_{h(t)}^* \alpha_{h(s)}\big)\, d\eta(s)\, d\bar{\eta}(t). \end{aligned} \tag{7.20}$$

Proof. In this example we have

$$\begin{aligned} \big\|\omega(h,\cdot)\big\|_{H^*\otimes \mathbf{C}}^2 &= \sum_{j=1}^{\infty}\sum_{a=1}^{d} \big\|\omega(h, l_j u_a)\big\|_{\mathbf{C}}^2 \\ &= \sum_{j=1}^{\infty}\sum_{a=1}^{d} \left\| \int_0^1 \alpha\big(h(s), l_j(s) u_a\big)\, d\eta(s) \right\|_{\mathbf{C}}^2 \\ &= \sum_{j=1}^{\infty}\sum_{a=1}^{d} \left\langle \int_0^1 \alpha\big(h(s), l_j(s) u_a\big)\, d\eta(s), \int_0^1 \alpha\big(h(t), l_j(t) u_a\big)\, d\eta(t) \right\rangle_{\mathbf{C}} \\ &= \int_0^1 d\eta(s) \int_0^1 d\bar{\eta}(t) (s\wedge t) \sum_{a=1}^{d} \langle \alpha_{h(s)} u_a, \alpha_{h(t)} u_a\rangle_{\mathbf{C}} \\ &= \int_{[0,1]^2} s\wedge t \big[\operatorname{tr}\big(\alpha_{h(t)}^* \alpha_{h(s)}\big)\big]\, d\eta(s)\, d\bar{\eta}(t). \end{aligned}$$

Using this identity along with Eq. (3.40) in Eq. (7.10) with $\alpha_{\mathbb{F}} = \alpha_{\mathbb{C}} = 2$ implies Eq. (7.20). □

8. Heat inequalities

8.1. Infinite-dimensional Radon–Nikodym derivative estimates

Recall from Theorem 6.1 and Corollary 6.2, we have already shown that $\nu_T \circ l_h^{-1}$ and $\nu_T \circ r_h^{-1}$ are absolutely continuous to ν_T for all $h \in G_{\mathrm{CM}}$ and $T > 0$. These results were based on the path space quasi-invariance formula given in Theorem 5.2. However, in light of the results in Malliavin [40] it is surprising that Theorem 5.2 holds at all and we do not expect it to extend to many other situations. Therefore, it is instructive to give an independent proof of Theorem 6.1 and Corollary 6.2 which will work for a much larger class of examples. The alternative proof has the added advantage of giving detailed size estimates on the resulting Radon–Nikodym derivatives.

Theorem 8.1. *For all $h \in G_{\mathrm{CM}}$ and $T > 0$, $\nu_T \circ l_h^{-1}$ and $\nu_T \circ r_h^{-1}$ are absolutely continuous to ν_T. Let $Z_h^{\mathrm{l}} := \frac{d(\nu_T \circ l_h^{-1})}{d\nu_T}$ and $Z_h^{\mathrm{r}} := \frac{d(\nu_T \circ r_h^{-1})}{d\nu_T}$ be the respective Radon–Nikodym derivatives, $k(\omega)$ is given in Eq. (7.6), and*

$$c(t) := \frac{t}{e^t - 1} \quad \text{for all } t \in \mathbb{R}$$

with the convention that $c(0) = 1$. Then for all $1 \leqslant p < \infty$, Z_h^{l} and Z_h^{r} are both in $L^p(\nu_T)$ and satisfy the estimate

$$\|Z_h^*\|_{L^p(\nu_T)} \leqslant \exp\left(\frac{c(k(\omega)T)(p-1)}{2T} d_{G_{\mathrm{CM}}}^2(\mathbf{e}, h)\right), \tag{8.1}$$

where $* = \mathrm{l}$ or $* = \mathrm{r}$.

Proof. The proof of this theorem is an application of Theorem 7.3 and Corollary 7.4 in [18] on quasi-invariance of the heat kernel measures for inductive limits of finite-dimensional Lie groups. In applying these results the reader should take: $G_0 = G_{\mathrm{CM}}$, $A = \mathrm{Proj}(W)$, $s_P := \pi_P$, $\nu_P = \mathrm{Law}(g_P(T))$, and $\nu = \nu_T = \mathrm{Law}(g(T))$. We now verify that the hypotheses [18, Theorem 7.3] are satisfied. These assumptions include a denseness condition on the inductive limit group, consistency of the heat kernel measures on finite-dimensional Lie groups, uniform bound on the Ricci curvature, and finally that the length of a path in the inductive limit group can be approximated by the lengths of paths in finite-dimensional groups.

(1) By Proposition 3.10, $\bigcup_{P \in \mathrm{Proj}(W)} G_P$ is a dense subgroup of G_{CM}.
(2) From Lemma 4.7, for any $\{P_n\}_{n=1}^{\infty} \subset \mathrm{Proj}(W)$ with $P_n|_H \uparrow I_H$ and $f \in BC(G, \mathbb{R})$ (the bounded continuous maps from G to $\mathbb{R}$), we have

$$\int_G f\, d\nu = \lim_{n\to\infty} \int_{G_{P_n}} f|_{G_{P_n}}\, d\nu_{P_n}.$$

(3) Corollary 7.3 shows that $\mathrm{Ric}_P \geqslant k(\omega) g_P$ for all $P \in \mathrm{Proj}(W)$.

(4) Lastly we have to verify that for any $P_0 \in \mathrm{Proj}(W)$, and $k \in C^1([0,1], G_{\mathrm{CM}})$ with $k(0) = e$, there exists an increasing sequence, $\{P_n\}_{n=1}^{\infty} \subset \mathrm{Proj}(W)$ such that $P_0 \subset P_n$, $P_n \uparrow I$ on H, and

$$\ell_{G_{\mathrm{CM}}}(k) = \lim_{n\to\infty} \ell_{G_{P_n}}(\pi_n \circ k), \tag{8.2}$$

where $\pi_n := \pi_{P_n}$ and $\ell_{G_{\mathrm{CM}}}(k)$ is the length of k (see Notation 3.9 with $T = 1$). However, with $k(t) = (A(t), c(t))$, using the dominated convergence theorem applied to the identity (see Eq. (3.21))

$$\begin{aligned}\ell_{G_{P_n}}(\pi_n \circ k) &= \int_0^1 \left\| \pi_n \dot{k}(t) - \frac{1}{2}\left[\pi_n k(t), \pi_n \dot{k}(t)\right] \right\|_{\mathfrak{g}_{\mathrm{CM}}} dt \\ &= \int_0^1 \sqrt{\left\| P_n \dot{A}(t) \right\|_H^2 + \left\| \dot{c}(t) - \frac{1}{2}\omega\left(P_n A(t), P_n \dot{A}(t)\right) \right\|_{\mathbf{C}}^2}\, dt\end{aligned}$$

shows Eq. (8.2) holds for any such choice of $P_n|_H \uparrow I_H$ with $P_0 \subset P_n \in \mathrm{Proj}(W)$. □

Remark 8.2. In the case of infinite-dimensional matrix groups three out of four assumptions hold as has been shown in [24]. The condition that fails is the uniform bounds on the Ricci curvature which is one of the main results in [25].

8.2. Logarithmic Sobolev inequality

Theorem 8.3. *Let $(\mathcal{E}_T, \mathcal{D}(\mathcal{E}_T))$ be the closed Dirichlet form in Corollary* 6.7 *and $k(\omega)$ be as in Eq.* (7.6)*. Then for all real-valued $f \in \mathcal{D}(\mathcal{E}_T)$, the following logarithmic Sobolev inequality holds*

$$\int_G \left(f^2 \ln f^2\right) d\nu_T \leqslant 2\frac{1 - e^{-k(\omega)T}}{k(\omega)} \mathcal{E}_T(f, f) + \int_G f^2\, d\nu_T \cdot \ln \int_G f^2\, d\nu_T, \tag{8.3}$$

where $\nu_T = \mathrm{Law}(g(T))$ is the heat kernel measure on G as in Definition 4.2.

Proof. Let $f : G \to \mathbb{R}$ be a cylinder polynomial as in Definition 6.6. Following the method of Bakry and Ledoux applied to G_P (see [22, Theorem 2.9] for the case needed here) shows

$$\begin{aligned}\mathbb{E}\left[\left(f^2 \log f^2\right)\left(g_P(T)\right)\right] &\leqslant 2\frac{1 - e^{-k_P(\omega)T}}{k_P(\omega)} \mathbb{E}\left\| \left(\operatorname{grad}^P f\right)\left(g_P(T)\right) \right\|_{G_P}^2 \\ &\quad + \mathbb{E}\left[f^2\left(g_P(T)\right)\right] \log \mathbb{E}\left[f^2\left(g_P(T)\right)\right]\end{aligned} \tag{8.4}$$

where $k_P(\omega)$ is as in Eq. (7.6). Since the function, $x \to x^{-1}(1 - e^{-x})$, is decreasing and $k(\omega) \leqslant k_P(\omega)$ for all $P \in \mathrm{Proj}(W)$, Eq. (8.4) also holds with $k_P(\omega)$ replaced by $k(\omega)$. With this observation along with Lemma 4.7, we may pass to the limit at $P \uparrow I$ in Eq. (8.4) to find

$$\mathbb{E}\big[\big(f^2\log f^2\big)\big(g(T)\big)\big] \leqslant 2\frac{1-e^{-k(\omega)T}}{k(\omega)}\mathbb{E}\big|\operatorname{grad} f\big(g(T)\big)\big|^2 + \mathbb{E}\big[f^2\big(g(T)\big)\big]\log\mathbb{E}\big[f^2\big(g(T)\big)\big].$$

This is equivalent to Eq. (8.3) when f is a cylinder polynomial. The result for general $f \in \mathcal{D}(\mathcal{E}_T)$ then holds by a standard (and elementary) limiting argument—see the end of Example 2.7 in [29]. □

9. Future directions

In this last section we wish to speculate on a number of ways that the results in this paper might be extended.

(1) It would be interesting to see what happens if we set B_0 to be identically zero so that $g(t)$ in Eq. (4.2) becomes

$$g(t) = \left(B(t), \frac{1}{2}\int_0^t \omega\big(B(\tau), \dot{B}(\tau)\big)\,d\tau\right). \tag{9.1}$$

The generator now is $L = \frac{1}{2}\sum_{k=1}^{\infty} \widetilde{(e_k, 0)}^2$ and if $\omega(\mathfrak{g}_{CM} \times \mathfrak{g}_{CM})$ is a total subset of $\mathbf{C}$, L would satisfy Hörmander's condition for hypoellipticity. If $\dim H$ were finite, it would follow that the heat kernel measure, ν_T, is a smooth positive measure and hence quasi-invariant. When $\dim H$ is infinite we do not know if ν_T is still quasi-invariant. Certainly both proofs which were given above when B_0 was not zero now break down.

(2) It should be possible to remove the restriction on $\mathbf{C}$ being finite-dimensional, i.e. we expect much of what we have done to go through when $\mathbf{C}$ is a separable Hilbert space. In doing so one would have to modify the finite-dimensional approximations used in the theory to truncate $\mathbf{C}$ as well.

(3) It should be possible to widen the class of admissible ωs substantially. The idea is to assume that ω is only defined from $H \times H \to \mathbf{C}$ such that $\|\omega\|_2 < \infty$. Under this relaxed assumption, we will no longer have a group structure on $G := W \times \mathbf{C}$. Nevertheless, with a little work one can still make sense of Brownian motion process defined in Definition 4.2 by letting

$$\int_0^t \omega\big(B(\tau), dB(\tau)\big) := L^2\text{-}\lim_{n\to\infty}\int_0^t \omega\big(P_nB(\tau), dP_nB(\tau)\big). \tag{9.2}$$

In fact, using Nelson's hypercontractivity and the fact that

$$\int_0^t \omega\big(P_nB(\tau), dP_nB(\tau)\big)$$

is in the second homogeneous chaos subspace, the convergence in Eq. (9.2) is in L^p for all $p \in [1, \infty)$. In this setting we expect the path space quasi-invariance results of Section 5 to remain valid. Similarly, as the lower bound on the Ricci curvature only depends on $\omega|_{H\times H}$, we expect the results of Section 8.1 to go through as well. As a consequence, G should carry a measurable left and right actions by element of G_{CM} and these actions should leave the heat kernel measures (end point distributions of the Brownian motion on G) quasi-invariant. One might call the resulting structure a *quasigroup*. Unfortunately, this term has already been used in abstract algebra.

(4) We also expect that level of non-commutativity of G may be increased. To be more precise, under suitable hypotheses, it should be possible to handle more general graded nilpotent Lie groups. However, when the level of nilpotency of G is increased, there will likely be trouble with the path space quasi-invariance in Section 5. Nevertheless, the methods of Section 8.1 should survive and therefore we still expect the heat kernel measure to be quasi-invariant.

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Appendix A. Wiener space results

The well-known material presented in this appendix may be (mostly) found in the books [36] and [8]. In particular, the following theorem is based in part on Lemma 2.4.1 in [8, p. 59], and Theorem 3.9.6 in [8, p. 138].

Theorem A.1. *Let* $(X, \mathcal{B}_X, \mu)$ *be a Gaussian measure space as in Definition* 2.1. *Then*

(1) $(H, \|\cdot\|_H)$ *is a normed space such that*

$$\|h\|_X \leqslant \sqrt{C_2}\|h\|_H \quad \text{for all } h \in H, \tag{A.1}$$

where C_2 *is as in* (2.2).

(2) *Let* K *be the closure of* X^* *in* $\operatorname{Re} L^2(\mu)$ *and for* $f \in K$ *let*

$$\iota f := h_f := \int_X x f(x)\, d\mu(x) \in X,$$

where the integral is to be interpreted as a Bochner integral. Then $\iota(K) = H$ *and* $\iota : K \to H$ *is an isometric isomorphism of real Banach spaces. Since* K *is a real Hilbert space it follows that* $\|\cdot\|_H$ *is a Hilbertian norm on* H.

(3) H *is a separable Hilbert space and*

$$(\iota u, h)_H = u(h) \quad \text{for all } u \in X^* \text{ and } h \in H. \tag{A.2}$$

(4) *The Cameron–Martin space,* H*, is dense in* X.

(5) *The quadratic form* q *may be computed as*

$$q(u, v) = \sum_{i=1}^{\infty} u(e_i) v(e_i) \tag{A.3}$$

where $\{e_i\}_{i=1}^{\infty}$ *is any orthonormal basis for* H.

Notice that by item (1) $H \overset{i}{\hookrightarrow} X$ *is continuous and hence so is* $X^* \overset{i^{\mathrm{tr}}}{\hookrightarrow} H^* \cong H = (\cdot,\cdot)_{H^*}$. *Eq.* (A.3) *asserts that*

$$q = (\cdot,\cdot)_{H^*}|_{X^*\times X^*}.$$

Proof. (1) Using Eq. (2.6) we find

$$\|h\|_X = \sup_{u\in X^*\setminus\{0\}} \frac{|u(h)|}{\|u\|_{X^*}} \leqslant \sup_{u\in X^*\setminus\{0\}} \frac{|u(h)|}{\sqrt{q(u,u)/C_2}} \leqslant \sqrt{C_2}\|h\|_H,$$

and hence if $\|h\|_H = 0$ then $\|h\|_X = 0$ and so $h = 0$. If $h, k \in H$, then for all $u \in X^*$, $|u(h)| \leqslant \|h\|_H \sqrt{q(u)}$ and $|u(k)| \leqslant \|k\|_H \sqrt{q(u)}$ so that

$$\left|u(h+k)\right| \leqslant \left|u(h)\right| + \left|u(k)\right| \leqslant \left(\|h\|_H + \|k\|_H\right)\sqrt{q(u)}.$$

This shows $h+k \in H$ and $\|h+k\|_H \leqslant \|h\|_H + \|k\|_H$. Similarly, if $\lambda \in \mathbb{R}$ and $h \in H$, then $\lambda h \in H$ and $\|\lambda h\|_H = |\lambda|\|h\|_H$. Therefore H is a subspace of W and $(H, \|\cdot\|_H)$ is a normed space.

(2) For $f \in K$ and $u \in X^*$

$$u(\iota f) = u\left(\int_X x f(x)\, d\mu(x)\right) = \int_X u(x) f(x)\, d\mu(x) \tag{A.4}$$

and hence

$$\left|u(\iota f)\right| \leqslant \|u\|_{L^2(\mu)} \|f\|_{L^2(\mu)} = \sqrt{q(u)}\|f\|_K$$

which shows that $\iota f \in H$ and $\|\iota f\|_H \leqslant \|f\|_K$. Moreover, by choosing $u_n \in X^*$ such that $L^2(\mu)\text{-}\lim_{n\to\infty} u_n = f$, we find

$$\lim_{n\to\infty} \frac{|u_n(\iota f)|}{\sqrt{q(u_n)}} = \lim_{n\to\infty} \frac{|\int_X u_n(x) f(x)\, d\mu(x)|}{\|u_n\|_{L^2(\mu)}} = \frac{\|f\|^2_{L^2(\mu)}}{\|f\|_{L^2(\mu)}} = \|f\|_{L^2(\mu)}$$

from which it follows $\|\iota f\|_H = \|f\|_K$. So we have shown that $\iota : K \to H$ is an isometry. Let us now show that $\iota(K) = H$. Given $h \in H$ and $u \in X^*$ let $\hat{h}(u) = u(h)$. Since

$$\left|\hat{h}(u)\right| \leqslant \left|u(h)\right| \leqslant \sqrt{q(u)}\|h\|_H = \|u\|_{L^2(\mu)}\|h\|_H = \|u\|_K\|h\|_H$$

the functional $\hat{h}$ extends continuously to K. We will continue to denote this extension by $\hat{h} \in K^*$. Since K is a Hilbert space, there exists $f \in K$ such that

$$\hat{h}(u) = \int_X f(x) u(x)\, d\mu(x)$$

for all $u \in X^*$ (and in fact all $u \in K$). Thus we have, for all $u \in X^*$, that

$$u(h) = \int_X u(x) f(x)\, d\mu(x) = u\left(\int_X x f(x)\, d\mu(x)\right) = u(\iota f).$$

From this equation we conclude that $h = \iota f$ and hence $\iota(K) = H$.

(3) H is a separable since it is unitarily equivalent to $K \subset L^2(X, \mathcal{B}, \mu)$ and $L^2(X, \mathcal{B}, \mu)$ is separable. Suppose that $u \in X^*$, $f \in K$ and $h = \iota f \in H$. Then

$$\begin{aligned}(\iota u, h)_H &= (\iota u, \iota f)_H = (u, f)_K \\ &= \int_X u(x) f(x)\, d\mu(x) = u\left(\int_X x f(x)\, d\mu(x)\right) \\ &= u(\iota f) = u(h).\end{aligned}$$

(4) For sake of contradiction, if $H \subset X$ were not dense, then, by the Hahn–Banach theorem, there would exist $u \in X^* \setminus \{0\}$ such that $u(H) = 0$. However from Eq. (A.2), we would then have

$$q(u, u) = (\iota u, \iota u)_H = u(\iota u) = 0.$$

Because we have assumed that q to be an inner product on X^*, u must be zero contrary to u being in $X^* \setminus \{0\}$.

(5) Let $\{e_i\}_{i=1}^\infty$ be an orthonormal basis for H, then for $u, v \in X^*$,

$$\begin{aligned}q(u, v) = (u, v)_K = (\iota u, \iota v)_H &= \sum_{i=1}^\infty (\iota u, e_i)_H (e_i, \iota v)_H \\ &= \sum_{i=1}^\infty u(e_i) v(e_i)\end{aligned}$$

wherein the last equality we have again used Eq. (A.2). □

Appendix B. The Ricci tensor on a Lie group

In this appendix we recall a formula for the Ricci tensor relative to a left invariant Riemannian metric, $\langle \cdot, \cdot \rangle$, on any finite-dimensional Lie group, G. Let ∇ be the Levi-Civita covariant derivative on TG, for any $X \in \mathfrak{g}$ let $\tilde{X}(g) = l_{g*} X$ be the left-invariant vector field on G such that $\tilde{X}(e) = X$, and for $X, Y \in \mathfrak{g}$, let $D_X Y := \nabla_X \tilde{Y} \in \mathfrak{g}$. Since $\nabla_{\tilde{X}} \tilde{Y}$ is a left-invariant vector field and $(\nabla_{\tilde{X}} \tilde{Y})(e) = \nabla_X \tilde{Y} = D_X Y$, we have the identity; $\nabla_{\tilde{X}} \tilde{Y} = \widetilde{D_X Y}$. Similarly the Ricci curvature tensor, Ric (and more generally the full curvature tensor) is invariant under left translations, i.e. $\mathrm{Ric}_g = l_{g^{-1}*} \mathrm{Ric}_e\, l_{g*}$ for all $g \in G$. Hence it suffices to compute the Ricci tensor at $e \in G$. We will abuse notation and simply write Ric for Ric_e.

Proposition B.1 *(The Ricci tensor on G). Continuing the notation above, for all $X, Y \in \mathfrak{g}$ we have*

$$D_X Y := \frac{1}{2}\big([X,Y] - \mathrm{ad}_X^* Y - \mathrm{ad}_Y^* X\big) \in \mathfrak{g}, \tag{B.1}$$

where ad_X^* denotes the adjoint of ad_X relative to $\langle \cdot,\cdot \rangle_e$. We also have,

$$\langle \mathrm{Ric}\, X, X\rangle = \mathrm{tr}(\mathrm{ad}_{\mathrm{ad}_X^* X}) - \frac{1}{2}\mathrm{tr}\big(\mathrm{ad}_X^2\big) + \frac{1}{4}\sum_{Y\in\Gamma}\big|\mathrm{ad}_Y^* X\big|^2 - \frac{1}{2}\sum_{Y\in\Gamma}|\mathrm{ad}_Y X|^2, \tag{B.2}$$

where $\Gamma \subset \mathfrak{g}$ is any orthonormal basis for $\mathfrak{g}$. In particular if $\mathfrak{g}$ is nilpotent then $\mathrm{tr}(\mathrm{ad}_{\mathrm{ad}_X^* X}) = 0$ *and* $\mathrm{tr}(\mathrm{ad}_X^2) = 0$ *and therefore Eq.* (B.2) *reduces to*

$$\langle \mathrm{Ric}\, X, X\rangle = \frac{1}{4}\sum_{Y\in\Gamma}\big|\mathrm{ad}_Y^* X\big|^2 - \frac{1}{2}\sum_{Y\in\Gamma}|\mathrm{ad}_Y X|^2 \geqslant -\frac{1}{2}\sum_{Y\in\Gamma}|\mathrm{ad}_Y X|^2. \tag{B.3}$$

These results may be found in [7], see Lemma 7.27, Theorem 7.30, and Corollary 7.33 for the computations of the Levi-Civita covariant derivative, the curvature tensor, and the Ricci curvature tensor, respectively.

Appendix C. Proof of Theorem 3.12

Before giving the proof of Theorem 3.12 it will be necessary to introduce the Carnot–Carathéodory distance function, δ, in this infinite-dimensional context.

Notation C.1. Let $T > 0$ and HC^1_{CM} denote the horizontal elements in C^1_{CM}, where $g \in C^1_{\mathrm{CM}}$ is *horizontal* iff $l_{g(s)^{-1}*} g'(s) \in H \times \{0\}$ for all s. We then define,

$$\delta(x,y) = \inf\big\{\ell_{G_{\mathrm{CM}}}(g)\colon\ g \in HC^1_{\mathrm{CM}} \text{ such that } g(0) = x \text{ and } g(T) = y\big\}$$

with the infimum of the empty set is taken to be infinite.

Observe that $\delta(x,y) \geqslant d_{\mathrm{CM}}(x,y)$ for all $x, y \in G_{\mathrm{CM}}$. The following theorem describes the behavior of δ.

Theorem C.2. *If $\{\omega(A,B)\colon\ A, B \in H\}$ is a total subset of $\mathbf{C}$, then there exists $c \in (0,1)$ such that*

$$c\big(\|A\|_H + \sqrt{\|a\|_{\mathbf{C}}}\big) \leqslant \delta\big(\mathbf{e},(A,a)\big) \leqslant c^{-1}\big(\|A\|_H + \sqrt{\|a\|_{\mathbf{C}}}\big) \quad \textit{for all } (A,a) \in \mathfrak{g}_{\mathrm{CM}}. \tag{C.1}$$

Proof. Our proof will be modeled on the standard proof of this result in the finite-dimensional context, see for example [42,50]. The only thing we must be careful of is to avoid using any compactness arguments.

For any left-invariant metric d (e.g. $d = \delta$ or $d = d_{\mathrm{CM}}$) on G_{CM} we have

$$d(\mathbf{e}, xy) \leqslant d(\mathbf{e}, x) + d(x, xy) = d(\mathbf{e}, x) + d(\mathbf{e}, y) \quad \text{for all } x, y \in G_{\mathrm{CM}}. \tag{C.2}$$

Given any path $g = (w, c) \in C^1_{\mathrm{CM}}$ joining $\mathbf{e}$ to (A, a), we have from Eq. (3.20) that

$$\ell_{G_{\mathrm{CM}}}(g) = \int_0^1 \sqrt{\|w'(s)\|_H^2 + \|c'(s) - \omega(w(s), w'(s))/2\|_{\mathbf{C}}^2}\, ds \geqslant \int_0^1 \|w'(s)\|_H\, ds \geqslant \|A\|_H$$

from which it follows that

$$\delta\big(\mathbf{e}, (A, a)\big) \geqslant d_{\mathrm{CM}}\big(\mathbf{e}, (A, 0)\big) \geqslant \|A\|_H. \tag{C.3}$$

Since the path $g(t) = (tA, 0)$ is horizontal and

$$\|A\|_H = \ell_{G_{\mathrm{CM}}}(g) \geqslant \delta\big(\mathbf{e}, (A, 0)\big) \geqslant d_{\mathrm{CM}}\big(\mathbf{e}, (A, 0)\big) \geqslant \|A\|_H$$

it follows that

$$\delta\big(\mathbf{e}, (A, 0)\big) = d\big(\mathbf{e}, (A, 0)\big) = \|A\|_H \quad \text{for all } A \in H. \tag{C.4}$$

Given $A, B \in H$, let $\xi(t) = A\cos t + B\sin t$ for $0 \leqslant t \leqslant 2\pi$ and

$$g(t) = \left(\xi(t) - A, \frac{1}{2}\int_0^t \omega\big(\xi(\tau) - A, \dot{\xi}(\tau)\big)\, d\tau\right)$$

so that $l_{g(t)^{-1}_*}\dot{g}(t) = (\xi(t), 0)$, $g(0) = \mathbf{e}$, and

$$g(2\pi) = \left(0, \frac{1}{2}\int_0^{2\pi} \omega\big(\xi(\tau), \dot{\xi}(\tau)\big)\, d\tau\right) = \left(0, \frac{1}{2}\int_0^{2\pi} \omega(A, B)\, d\tau\right) = \big(0, \pi\omega(A, B)\big).$$

From this one horizontal curve we may conclude that

$$\delta\big(\mathbf{e}, \big(0, \pi\omega(A, B)\big)\big) \leqslant \ell_{G_{\mathrm{CM}}}(g) = \int_0^{2\pi} \|-A\sin t + B\cos t\|_H\, dt \leqslant 2\pi\big(\|A\|_H + \|B\|_H\big). \tag{C.5}$$

Choose $\{A_\ell, B_\ell\}_{\ell=1}^d \subset H$ such that $\{\pi\omega(A_\ell, B_\ell)\}_{\ell=1}^d$ is a basis for $\mathbf{C}$. Let $\{\varepsilon^\ell\}_{\ell=1}^d$ be the corresponding dual basis. Hence for any $a \in \mathbf{C}$ we have

$$
\begin{aligned}
\delta\big(\mathbf{e},(0,a)\big) &= \delta\left(\mathbf{e},\prod_{\ell=1}^{d}\big(0,\varepsilon^{\ell}(a)\pi\omega(A_\ell,B_\ell)\big)\right)\\
&\leqslant \sum_{\ell=1}^{d}\delta\big(\mathbf{e},\big(0,\varepsilon^{\ell}(a)\pi\omega(A_\ell,B_\ell)\big)\big)\\
&= \sum_{\ell=1}^{d}\delta\Big(\mathbf{e},\Big(0,\pi\omega\Big(\operatorname{sgn}\big(\varepsilon^{\ell}(a)\big)\sqrt{\big|\varepsilon^{\ell}(a)\big|}A_\ell,\sqrt{\big|\varepsilon^{\ell}(a)\big|}B_\ell\Big)\Big)\Big)\\
&\leqslant 2\pi\sum_{\ell=1}^{d}\Big(\Big\|\sqrt{\big|\varepsilon^{\ell}(a)\big|}A_\ell\Big\|_H+\Big\|\sqrt{\big|\varepsilon^{\ell}(a)\big|}B_\ell\Big\|_H\Big),
\end{aligned}
$$

wherein we have used Eq. (C.2) for the first inequality and Eq. (C.5) for the second inequality. It now follows by simple estimates that

$$
\delta\big(\mathbf{e},(0,a)\big)\leqslant C_1\sum_{\ell=1}^{d}\sqrt{\big|\varepsilon^{\ell}(a)\big|}\leqslant C_2\sqrt{\sum_{\ell=1}^{d}\big|\varepsilon^{\ell}(a)\big|}\leqslant C(\omega)\sqrt{\|a\|_{\mathbf{C}}} \tag{C.6}
$$

for some constants $C_1\leqslant C_2\leqslant C(\omega)<\infty$. Combining Eqs. (C.2), (C.4), and (C.6) gives,

$$
\begin{aligned}
\delta\big(\mathbf{e},(A,a)\big) &= \delta\big(\mathbf{e},(A,0)(0,a)\big)\\
&\leqslant \delta\big(\mathbf{e},(A,0)\big)+\delta\big(\mathbf{e},(0,a)\big)\leqslant \|A\|_H+C(\omega)\sqrt{\|a\|_{\mathbf{C}}}.
\end{aligned} \tag{C.7}
$$

To prove the analogous lower bound we will make use of the dilation homomorphisms defined for each $\lambda>0$ by $\varphi_\lambda(w,c)=(\lambda w,\lambda^2 c)$ for all $(w,c)\in\mathfrak{g}_{\mathrm{CM}}=G_{\mathrm{CM}}$. One easily verifies that φ_λ is both a Lie algebra homomorphism on $\mathfrak{g}_{\mathrm{CM}}$ and a group homomorphism on G_{CM}. Using the homomorphism property it follows that

$$
l_{\varphi_\lambda(g(t))_*^{-1}}\frac{d}{dt}\varphi_\lambda\big(g(t)\big)=\varphi_\lambda\big(l_{g(t)_*^{-1}}\dot g(t)\big)
$$

and consequently; if g is any horizontal curve, then $\varphi_\lambda\circ g$ is again horizontal and $\ell_{G_{\mathrm{CM}}}(\varphi_\lambda\circ g)=\lambda\ell_{G_{\mathrm{CM}}}(g)$. From these observations we may conclude that

$$
\delta\big(\varphi_\lambda(x),\varphi_\lambda(y)\big)=\lambda\delta(x,y)\quad\text{for all } x,y\in G_{\mathrm{CM}}. \tag{C.8}
$$

By Proposition 3.10, we know there exists $\varepsilon>0$ and $K<\infty$ such that

$$
K\delta(\mathbf{e},x)\geqslant K d_{G_{\mathrm{CM}}}(\mathbf{e},x)\geqslant\|x\|_{\mathfrak{g}_{\mathrm{CM}}}\quad\text{whenever } \|x\|_{\mathfrak{g}_{\mathrm{CM}}}\leqslant\varepsilon. \tag{C.9}
$$

For arbitrary $x=(A,a)\in G_{\mathrm{CM}}$, choose $\lambda>0$ such that

$$
\varepsilon^2=\big\|\varphi_\lambda(x)\big\|^2=\lambda^2\|A\|_H^2+\lambda^4\|a\|_{\mathbf{C}}^2,
$$

i.e.

$$\lambda^2 = \frac{\sqrt{\|A\|_H^4 + 4\|a\|_{\mathbf{C}}^2\varepsilon^2} - \|A\|_H^2}{2\|a\|_{\mathbf{C}}^2}.$$

It then follows from Eqs. (C.8) and (C.9) that $\lambda K\delta(\mathbf{e}, x) = K\delta(\mathbf{e}, \varphi_\lambda(x)) \geqslant \varepsilon$, i.e.

$$\delta^2(\mathbf{e}, x) \geqslant \frac{\varepsilon^2}{K^2\lambda^2} = 2\frac{\varepsilon^2}{K^2}\frac{\|a\|_{\mathbf{C}}^2}{\sqrt{\|A\|_H^4 + 4\|a\|_{\mathbf{C}}^2\varepsilon^2} - \|A\|_H^2}$$
$$= 2\frac{\varepsilon^2\|a\|_{\mathbf{C}}^2}{K^2\|A\|_H^2}\frac{1}{\sqrt{1 + \frac{4\|a\|_{\mathbf{C}}^2\varepsilon^2}{\|A\|_H^4}} - 1}. \tag{C.10}$$

Since $\sqrt{1+x} - 1 \leqslant \min(x/2, \sqrt{x})$ we have

$$\frac{1}{\sqrt{1+x} - 1} \geqslant \max\left(\frac{2}{x}, \frac{1}{\sqrt{x}}\right) \geqslant \frac{1}{x} + \frac{1}{2\sqrt{x}}.$$

Using this estimate with $x = 4\|a\|_{\mathbf{C}}^2\|A\|_H^{-4}\varepsilon^2$ in Eq. (C.10) shows

$$\delta^2(\mathbf{e}, x) \geqslant 2\frac{\varepsilon^2\|a\|_{\mathbf{C}}^2}{K^2\|A\|_H^2}\left(\frac{\|A\|_H^4}{4\|a\|_{\mathbf{C}}^2\varepsilon^2} + \frac{\|A\|_H^2}{4\varepsilon\|a\|_{\mathbf{C}}}\right) = \frac{1}{2K^2}\left(\|A\|_H^2 + \varepsilon\|a\|_{\mathbf{C}}\right),$$

which implies the lower bound in Eq. (C.1). □

We are now ready to give the proof of Theorem 3.12.

C.1. Proof of Theorem 3.12

Proof. The first assertion in Eq. (3.24) of Theorem 3.12 follows from Theorem C.2 and the previously observed fact that $d_{\mathrm{CM}} \leqslant \delta$. To prove Eq. (3.25), let $\varepsilon_0 < \varepsilon/2$ where $\varepsilon > 0$ is as in Proposition 3.10. Then according to that proposition, if $d_{\mathrm{CM}}(\mathbf{e}, x) \leqslant \varepsilon_0$ then $\|x\|_{\mathfrak{g}_{\mathrm{CM}}} \leqslant K d_{\mathrm{CM}}(\mathbf{e}, x) \leqslant K\varepsilon_0$. So if $x = (A, a)$, we have $\|A\|_H \leqslant K\varepsilon_0$ and $\|a\|_{\mathbf{C}} \leqslant K\varepsilon_0$ and hence by Theorem C.2, $\delta(\mathbf{e}, x) \leqslant c^{-1}(K\varepsilon_0 + \sqrt{K\varepsilon_0})$. This implies that

$$M(\varepsilon_0) := \sup\{\delta(\mathbf{e}, x)\colon x \ni d_{\mathrm{CM}}(\mathbf{e}, x) \leqslant \varepsilon_0\} \leqslant c^{-1}(K\varepsilon_0 + \sqrt{K\varepsilon_0}) < \infty. \tag{C.11}$$

Now suppose that $x \in G_{\mathrm{CM}}$ with $d_{\mathrm{CM}}(\mathbf{e}, x) \geqslant \varepsilon_0$. Choose a curve, $g \in C^1_{\mathrm{CM}}$ such that $g(0) = \mathbf{e}$, $g(1) = x$, and $\ell_{G_{\mathrm{CM}}}(g) \leqslant d_{\mathrm{CM}}(\mathbf{e}, x) + \varepsilon_0/4$. Also choose $\varepsilon_1 \in (\varepsilon_0/2, \varepsilon_0]$ such that $\ell_{G_{\mathrm{CM}}}(g) = n\varepsilon_1$ with $n \in \mathbb{N}$ and let $0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$ be a partition of [0, 1] such that $\ell_{G_{\mathrm{CM}}}(g|_{[t_{i-1}, t_i]}) = \varepsilon_1$ for $i = 1, 2, \ldots, n$. If $x_i := g(t_i)$ for $i = 0, \ldots, n$, then $\varepsilon_0 \geqslant \varepsilon_1 = \ell_{G_{\mathrm{CM}}}(g|_{[t_{i-1}, t_i]}) \geqslant d_{\mathrm{CM}}(x_{i-1}, x_i)$ and therefore from Eq. (C.11) and the left invariance of d_{CM} and δ we have $1 \geqslant M(\varepsilon_0)^{-1}\delta(x_{i-1}, x_i)$ for $i = 1, 2, \ldots, n$. Hence we may conclude that

$$2d_{\mathrm{CM}}(\mathbf{e},x) \geqslant d_{\mathrm{CM}}(\mathbf{e},x)+\varepsilon_0/4 \geqslant \ell_{G_{\mathrm{CM}}}(g)=\varepsilon_1 n$$
$$\geqslant \varepsilon_1 \sum_{i=1}^{n} M(\varepsilon_0)^{-1}\delta(x_{i-1},x_i) \geqslant \frac{\varepsilon_0}{2} M(\varepsilon_0)^{-1}\delta(\mathbf{e},x).$$

Combining this estimate with the lower bound in Eq. (C.1) shows Eq. (3.25) holds for all ε_0 sufficiently small which is enough to complete the proof. □

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