Talk 2: “Quantized Yang-Mills (d=2) and the Segal-Bargmann-Hall Transform”

Bruce Driver

Department of Mathematics, 0112
University of California at San Diego, USA

http://math.ucsd.edu/∼bdriver

Nelder Talk 2.

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Gaussian Measures on Hilbert spaces

Goal: Given a Hilbert space H, we would ideally like to define a probability measure μ on $\mathcal{B}(H)$ such that

$$\hat{\mu}(h) := \int_H e^{i(\lambda,x)} d\mu(x) = e^{-\frac{1}{2}\|\lambda\|^2} \text{ for all } \lambda \in H \tag{1}$$

so that, informally,

$$d\mu(x) = \frac{1}{Z} e^{-\frac{1}{2}|x|_H^2} \mathcal{D}x. \tag{2}$$

The next proposition shows that this is impossible when $\dim(H) = \infty$.

Proposition 1. *Suppose that H is an infinite dimensional Hilbert space. Then there is no probability measure μ on the Borel σ – algebra, $\mathcal{B} = \mathcal{B}(H)$, such that Eq. (1) holds.*

Proof: Suppose such a Gaussian measure were to exist. If $\{e_i\}_{i=1}^{\infty}$ is an ON basis for H, then $\{\langle e_i, \cdot\rangle\}_{i=1}^{\infty}$ would be i.i.d. normal random variables. By SSLN,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \langle e_i, \cdot\rangle^2 = 1 \ \mu-\text{a.s.}$$

which would imply

$$\infty > \|x\|^2 = \sum_{i=1}^{\infty} \langle e_i, x \rangle^2 = \infty \text{ a.s.}$$

Q.E.D.

Moral: The measure μ must be defined on a larger space. This is somewhat analogous to trying to define Lebesgue measure on the rational numbers. In each case the measure can only be defined on a certain completion of the naive initial space.

A Non-Technicality

Theorem 2. *Let $\mathbb{Q}$ be the rational numbers.*

1. *There is no translation invariant **measure** (m) on $\mathbb{Q}$ which is finite on bounded sets.*
2. *Similarly there is no **measure** (m) on $\mathbb{Q}$ such that $m(\{x \in \mathbb{Q} : a < x < b\}) = b - a$.*

Proof: In either case one shows that $m(\{r\}) = 0$ and then by countable additivity

$$m(\mathbb{Q}) = \sum_{r \in \mathbb{Q}} m(\{r\}) = 0.$$

For example if m existed as in item 2., then $m(\{r\}) \leq b - a$ for any choice of $a < r < b$ which can only be if $m(\{r\}) = 0$. Q.E.D.

MORAL: To construct desirable countably additive measures the underlying set must be sufficiently "big."

Measures on Hilbert Spaces

Theorem 3. *Suppose that H and K are separable Hilbert spaces, H is a dense subspace of K, and the inclusion map, $i : H \to K$ is continuous. Then there exists a Gaussian measure, ν, on K such that*

$$\int_K e^{\lambda(x)} d\nu(x) = \exp\left(\frac{1}{2}(\lambda, \lambda)_{H^*}\right) \text{ for all } \lambda \in K^* \subset H^* \tag{3}$$

iff $i : H \to K$ is Hilbert Schmidt. Recalling the Hilbert Schmidt norm of i and its adjoint, i^*, are the same, the following conditions are equivalent;

1. *$i : H \to K$ is Hilbert Schmidt,*
2. *$i^* : K \to H$ is Hilbert Schmidt,*
3. *$\operatorname{tr}(i\, i^*) < \infty$*
4. *$\operatorname{tr}(i^* i) < \infty$.*

Proof: We only prove here; if $i : H \to K$ is Hilbert Schmidt, then there exists a measure ν on K such that Eq. (3) holds. For the converse direction, see [Bogachev, 1998, Da Prato & Zabczyk, 1992, Kuo, 1975].

- $A := i^* i : H \to H$, is a self-adjoint trace class operator.

- By the spectral theorem, there exists an orthonormal basis, $\{e_j\}_{j=1}^{\infty}$ for H such that $Ae_j = a_j e_j$ with $a_j > 0$ and $\sum_{j=1}^{\infty} a_j < \infty$.

- $(e_j, e_k)_K = (ie_j, ie_k)_K = (i^* ie_j, e_k)_H = (Ae_j, e_k)_H = a_j \delta_{jk}$.

- Let $\{N_j\}_{j=1}^{\infty}$ be i.i.d. standard normal random variables and set

$$S := \sum_{j=1}^{\infty} N_j e_j.$$

- Notice that

$$\mathbb{E}\left[\|S\|_K^2\right] = \sum_{j=1}^{\infty} \|e_j\|_K^2 = \sum_{j=1}^{\infty} a_j < \infty$$

- Now take $\nu = \mathrm{Law}\,(S)$.

Q.E.D.

Wiener Measure Example

Example 1 (Wiener measure)**.** Let

$$H = \left\{ h : [0,T] \to \mathbb{R}^d | h(0) = 0 \text{ and } \langle h, h \rangle_H = \int_0^1 |h'(s)|^2 ds < \infty \right\}.$$

and take $K = L^2\left([0,T], \mathbb{R}^d\right)$. On then shows;

1. $(i^* f)(\tau) = \int_0^T \min(t,\tau) f(\tau)\, d\tau$
2. $\mathrm{tr}(i\, i^*) = d \cdot \int_0^T \min(t,t)\, dt = d \cdot T^2/2 < \infty.$

Euclidean Free Field

Definition 4. For $f \in C^\infty\left(\mathbb{T}^d\right)$, let

$$\|f\|_s^2 := \left\langle \left(-\Delta + m^2\right)^s f, f \right\rangle = \left\| \left(-\Delta + m^2\right)^{s/2} f \right\|_{L^2}^2$$

and set H_s be the closure inside of $\left[C^\infty\left(\mathbb{T}^d\right)\right]'$. [We normalize Lebesgue measure to have volume 1 on $\mathbb{T}^d$.]

Theorem 5. *The measure,*

$$d\mu(\varphi) = \frac{1}{Z} e^{-\int_{\mathbb{T}^d}\left[\frac{1}{2}|\nabla\varphi(x)|^2 + m^2\varphi^2(x)\right]dx} \mathcal{D}\varphi$$

exists on H_s iff $s < 1 - \frac{d}{2}$.

Proof: For $n \in \mathbb{Z}^d$, let $\chi_n(\theta) := e^{in\cdot\theta}$ for $\theta \in \mathbb{T}^d$. Then

$$\langle \chi_n, \chi_m \rangle_s = \left\langle \left(-\Delta + m^2\right)^s \chi_n, \chi_m \right\rangle = \left[|n|^2 + m^2\right]^s \delta_{mn}.$$

Therefore,

$$\left\{ \frac{\chi_n}{\sqrt{|n|^2 + m^2}} \right\}_{n \in \mathbb{Z}^d} \quad \text{is an ON basis for } H_1.$$

The result now follows since

$$\sum_{n\in\mathbb{Z}^d}\left\|\frac{\chi_n}{\sqrt{|n|^2+m^2}}\right\|_s^2=\sum_{n\in\mathbb{Z}^d}\frac{1}{\left(|n|^2+m^2\right)^{1-s}}$$

which is finite iff $2(1-s)>d \iff s<1-\frac{d}{2}$. Q.E.D.

Stochastic Quantization (Skipped)

Let V be a nice potential,

$$H = -\frac{1}{2}\Delta + V,$$
$$\lambda_0 = \inf \sigma(H) \text{ and } \Omega > 0 \ni H\Omega = \lambda_0 \Omega.$$

By making sense of

$$d\mu(\omega) = \frac{1}{Z} e^{-\int_{-\infty}^{\infty} \left\{ \frac{1}{2}(\omega'(s))^2 + V(\omega(s)) \right\} ds} \mathcal{D}\omega \tag{4}$$

We learn knowledge of Ω and $\widehat{H} := \Omega^{-1}(H - \lambda_0)\Omega$ via:

$$\int_W f(\omega(0)) d\mu(\omega) = \int \Omega^2(x) f(x) dx$$

$$\int_W f(\omega(0)) g(\omega(t)) d\mu(\omega) = \left(e^{t(H-\lambda_0)} \Omega f, \Omega g \right)_{L^2(dx)}$$
$$= \left(e^{t\widehat{H}} f, g, \right)_{L^2(\Omega^2 dx)}$$

Quantized Non-Linear Klein-Gordon Equation (Skipped)

$$\varphi_{tt} + (-\Delta + m^2)\varphi + \varphi^3 = 0$$

where $\varphi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$. Equivalently,

$$\varphi_{tt} = -\nabla V(\varphi)$$

where

$$V(\varphi) = \int_{\mathbb{R}^d} \left(\frac{1}{2} |\nabla \varphi|^2 + \frac{m^2}{2}\varphi^2 + \frac{1}{4}\varphi^4 \right) dx.$$

Quantization leads to the equation

$$\partial_t u(t,\varphi) = \frac{1}{2}\Delta_H u(t,\varphi) - V(\varphi)u(t,\varphi)$$

where $H := L^2(\mathbb{R}^d)$ with formal path integral quantization:

$$e^{T\left(\frac{1}{2}\Delta_H - V\right)} f(\varphi_o) = \frac{1}{Z_T} \int_{\varphi(0)=\varphi_0} e^{-\int_0^T \left[\frac{1}{2}\|\dot{\varphi}(t)\|_H^2 + V(\varphi(t))\right] dt} f(\varphi(T)) \mathcal{D}\varphi.$$

See Glimm and Jaffe's Book, 1987.

The appearance of infinities

For "interacting" quantum field theories one would like to make sense of

$$d\mu_v(\varphi) := \frac{1}{Z} e^{-\int_{\mathbb{T}^d} \left[\frac{1}{2}|\nabla\varphi(x)|^2 + m^2\varphi^2(x) + v(\varphi(x))\right] dx} \mathcal{D}\varphi$$

where $v(s)$ is a polynomial in s like $v(s) = s^4$. The obvious way to do this is to write,

$$\begin{aligned} d\mu_v(\varphi) &:= e^{-\int_{\mathbb{T}^d} v(\varphi(x)) dx} \frac{1}{Z} e^{-\int_{\mathbb{T}^d} \left[\frac{1}{2}|\nabla\varphi(x)|^2 + m^2\varphi^2(x)\right] dx} \mathcal{D}\varphi \\ &= \frac{1}{Z_v} e^{-\int_{\mathbb{T}^d} v(\varphi(x)) dx} \cdot d\mu_0(\varphi) \end{aligned}$$

where $d\mu_0(\varphi)$ is given in Theorem 5. However, μ_0 is only supported on $H_{1-\frac{d}{2}-\varepsilon}$ – a space of distributions and therefore $v(\varphi(x))$ is not well defined!

Path Integral
Quantized Yang-Mills Fields (Skipped)

- A $1,000,000 question, `http://www.claymath.org/millennium-problems`

- "… Quantum Yang-Mills theory is now the foundation of most of elementary particle theory, and its predictions have been tested at many experimental laboratories, but its mathematical foundation is still unclear. …"

- Roughly speaking one needs to make sense out of the path integral expressions above when $[0,T]$ is replaced by $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$:

$$d\mu(A)\text{``}=\text{''}\frac{1}{Z}\exp\left(-\frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}^3}\left|F^A\right|^2 dt\,dx\right)\mathcal{D}A, \tag{5}$$

- New problem: **gauge invariance.**

- We are going to discuss quantized Yang-Mills from the "Canonical quantization" point of view.

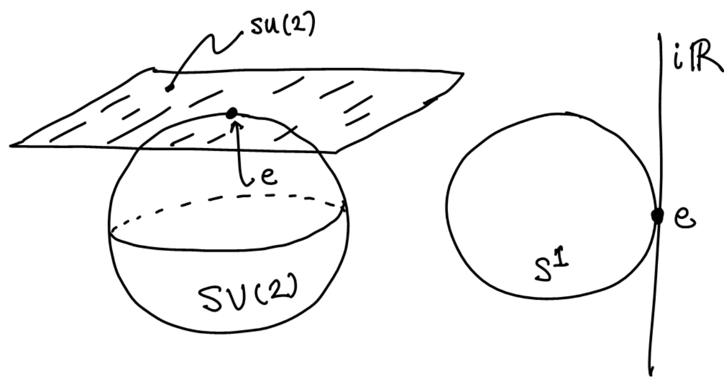
Gauge Theory Notation

- $K = SU(2)$ or S^1 or a compact Lie Group

$$SU(2) = \left\{ g := \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} : a, b \in \mathbb{C} \ni \ |a|^2 + |b|^2 = 1 \right\}$$

- $\mathfrak{k}$ = Lie(K), e.g. $\mathrm{Lie}(SU(2)) = su(2)$

$$su(2) = \left\{ A := \begin{bmatrix} i\alpha & -\bar{\beta} \\ \beta & -i\alpha \end{bmatrix} : \alpha \in \mathbb{R} \text{ and } \beta \in \mathbb{C} \right\}$$



- **Lie bracket:** $[A, B] = AB - BA =: ad_A B$

- $\langle A, B \rangle = -\mathrm{tr}(AB) = \mathrm{tr}(A^* B)$
  (a fixed Ad – K – invariant inner product)

- $M = \mathbb{R}^d$ or $T^d = \left(S^1\right)^d$.

- $\mathcal{A} = L^2(M, \mathfrak{k}^d)$ – the space of connection 1-forms.

- For $A \in \mathcal{A}$ and $1 \leq i, k \leq d$, let

$$\nabla_k^A := \partial_k + ad_{A_k} \text{ (covariant differential)}$$

and

$$F_{ki}^A := \partial_k A_i - \partial_i A_k + [A_k, A_i] \text{ (Curvature of } A)$$

Newton Form of the Y. M. Equations

Define the potential energy functional, $V(A)$, by

$$V(A) := \frac{1}{2}\int_{\mathbb{R}^d} \sum_{1\le j<k\le d} |F^A_{j,k}(x)|^2 dx.$$

Then the dynamics equation may be written in Newton form as

$$\ddot{A}(t) = -(\mathrm{grad}_{\mathcal{A}} V)(A).$$

The conserved energy is thus

$$\text{Energy}\left(A, \dot{A}\right) = \frac{1}{2}\left\|\dot{A}\right\|^2_{\mathcal{A}} + V(A). \qquad (6)$$

The weak form of the constraint equation,

$$0 = \nabla^A \cdot E = \sum_{k=1}^{d} \nabla^A_k E_k \text{ is}$$

$$0 = \left(E, \nabla^A h\right)_{\mathcal{A}} \ \forall\, h \in C_c^\infty(M, \mathfrak{k}).$$

Formal Quantization of the Y. M. – Equations

When $d = 3$, **"Quantize"** the Yang – Mills equations and show the resulting quantum – mechanical Hamiltonian has a mass gap. See $www.claymath.org$. Formally we have,

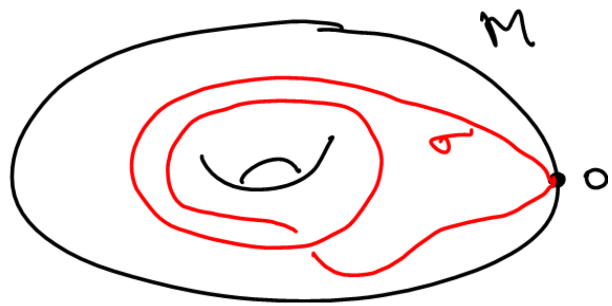
- Raw quantum Hilbert Space: $\mathbb{H} = L^2(\mathcal{A}, \text{“}\mathcal{D}A\text{”})$.
- Energy operator: $\hat{E} := -\frac{1}{2}\Delta_{\mathcal{A}} + M_V$ where

$$V(A) := \frac{1}{2}\int_{\mathbb{R}^d} \sum_{1 \le j < k \le d} |F^A_{j,k}(x)|^2 dx.$$

- This must all be restricted to the physical Hilbert space coming from the constraints.
- Some possible references of interest are; [Driver & Hall, 2000, Driver & Hall, 1999, Driver *et al.*, 2013, Hall, 2003, Hall, 2002, Hall, 2001, Hall, 1999] and the references therein.

Wilson Loop Variables

Let $\mathcal{L} = \mathcal{L}(M)$ loops on M based at $o \in M$.



Definition 6. Let $//^A(\sigma) \in K$ be **parallel translation along** $\sigma \in \mathcal{L}$, that is $//^A(\sigma) := //_1^A(\sigma)$, where

$$\frac{d}{dt}//_t^A(\sigma) + \sum_{i=1}^{d} \dot{\sigma}_i(t) A_i(\sigma(t)) //_t^A(\sigma) = 0 \text{ with } //_0^A(\sigma) = id.$$

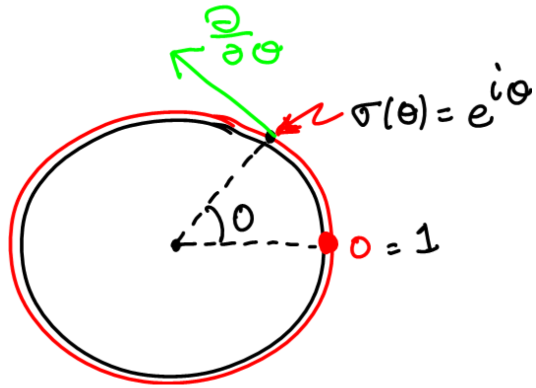
[Very ill defined unless $d = 1$!!]

- Physical quantum Hilbert Space

$$\mathbb{H}_{\text{physical}} = \left\{ F \in L^2(\mathcal{A}, \mathcal{D}A) : F = F\left(\left\{//^A(\sigma) : \sigma \in \mathcal{L}\right\}\right)\right\}$$

Restriction to $d = 1$

$S^1 = [0,1]/(0 \sim 1) \ni \theta$ and write $\partial_\theta = \frac{\partial}{\partial \theta}$



In this case,

- $\mathcal{A} = L^2(S^1, \mathfrak{k})$,
- $\mathcal{G}_0 = \{g \in H^1(S^1 \to K) : g(0) = g(1) = id \in K\}$,
- $A^g = Ad_{g^{-1}} A + g^{-1} g'$

- $\mathbb{H} =$"$L^2(\mathcal{A}, \mathcal{D}A)$"

- $\mathbb{H}_{\text{physical}} = \{F \in \mathbb{H} : F_\varphi(A) = \varphi(//_1(A)),\ \varphi : K \to \mathbb{C}\}$, where $//_\theta(A) \in K$ is the solution to
$$\frac{d}{d\theta}//_\theta(A) + A(\theta)//_\theta(A) = 0 \text{ with } //_0(A) = id \in K.$$
$//_1(A) \in K$ is the **holonomy** of A.

- $H = -\frac{1}{2}\Delta_{\mathcal{A}}$ $\quad$ (Quantum Hamiltonian)

Remark 7. $F^A \equiv 0$ when $d = 1$ and therefore, $V(A) \equiv 0$.

A Physics Idea

Theorem 8 (Heuristic: c.f. Witten 1991, CMP 141.). *Suppose K is simply connected and for φ let $F_\varphi(A) := \varphi(//_1(A))$, then*

$$\varphi \in L^2(K, d\mathit{Haar}) \to F_\varphi \in \mathbb{H}_{\mathit{physical}} \tag{7}$$

is a "Unitary" map which intertwines $\Delta_{\mathcal{A}}$ and Δ_K, i.e.

$$\Delta_{\mathcal{A}}[\varphi \circ //_1] = \Delta_{\mathcal{A}} F_\varphi = F_{\Delta_K \varphi} = (\Delta_K \varphi) \circ //_1. \tag{8}$$

Proof:

- Use $\langle \cdot, \cdot \rangle$ on $\mathfrak{k}$ to construct a bi-invariant metric on TK.
- Let $H(K)$ be the space of finite energy paths on K starting at $e \in K$.
- Equip $H(K)$ with the right invariant metric induced from the metric on
$$H(\mathfrak{k}) := \mathrm{Lie}(H(K)).$$
- The "Cartan Rolling Map, $\psi : \mathcal{A} \to H(K)$ defined by
$$\psi(A) := //_\cdot(A)$$
is an isometric isomorphism of Riemannian manifolds.

- Consequently we may “conclude” that ψ intertwines the Laplacian, $\Delta_{\mathcal{A}}$ on $\mathcal{A}$ with the Laplacian, $\Delta_{H(K)}$ on $H(K)$, i.e.

$$\Delta_{\mathcal{A}}(f \circ \psi) = \left(\Delta_{H(K)} f\right) \circ \psi. \tag{9}$$

When $f(g) = \varphi(g(1))$, one can show

$$\Delta_{H(K)} f(g) = (\Delta_K \varphi)(g(1))$$

and therefore Eq. (9) implies,

$$\Delta_{\mathcal{A}}(\varphi \circ //_1) = (\Delta_K \varphi) \circ //_1.$$

- Other geometric arguments show formally,

$$\int F(A) \mathcal{D}A = \int_K dk \int_{\psi_1^{-1}(k)} F(A) \, d\lambda_k(A),$$

where dk is Haar measure on K, λ_k is the formal Riemannian volume measure on $\psi_1^{-1}(k)$, and $\lambda_k\left(\psi_1^{-1}(k)\right)$ is constant independent of k.

Q.E.D.

A more precise Version of Theorem 8

- For $s > \frac{t}{2} > 0$ let

$$d\tilde{P}_s(A) = \frac{1}{Z_s} \exp\left(-\frac{1}{2s}|A|^2_{\mathcal{A}}\right)\mathcal{D}A \text{ and}$$

$$d\tilde{M}_{s,t}(A+iB) = \frac{1}{Z_{s,t}} \exp\left(-\frac{1}{2s-t}|A|^2_{\mathcal{A}} - \frac{1}{t}|B|^2_{\mathcal{A}}\right)\mathcal{D}A\mathcal{D}B.$$

- As we have seen one has to intpret these as Gaussian measures living on fattened up spaces, $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}_{\mathbb{C}} = \bar{\mathcal{A}} + i\bar{\mathcal{A}}$ respectively.
- "$\lim_{s\to\infty} d\tilde{P}_s(A) = c \cdot \mathcal{D}A$."

Theorem 9 (Segal- Bargmann)**.** *There exists an isometry*

$$S_t : L^2(\mathcal{A}, \tilde{P}_s) \to L^2(W(\mathfrak{k}_{\mathbb{C}}), M_{s,t})$$

such that

$$(S_t f)(c) = \int f_{\mathbb{C}}(c+a)dP_t(a) = (e^{\frac{t}{2}\triangle_{\mathcal{A}}} f)_a(c).$$

For all polynomial cylinder functions f. Moreover Ran(S_t) = closure of Holomorphic cylinder functions.

Main Theorem

Theorem 10 (Main Theorem, [Driver & Hall, 1999]). *Let*

$$\frac{d}{d\theta}//_{\theta}+A\left(\theta\right)//_{\theta}=0 \text{ with } //_{0}=Id$$

and

$$\frac{d}{d\theta}//_{\theta}^{\mathbb{C}}+\left(A\left(\theta\right)+iB\left(\theta\right)\right)//_{\theta}^{\mathbb{C}}=0 \text{ with } //_{0}^{\mathbb{C}}=Id$$

as "Stratonovich SDE's" relative to P_s and $M_{s,t}$ respectively. Then for all $f\in L^2(K,dx)$,

$$S_t\left[f(//_1)\right]=F(//_1^{\mathbb{C}})$$

where F is the unique Holomorphic function on $K_{\mathbb{C}}$ such that

$$F|_K=e^{\frac{t}{2}\triangle_K}f.$$

Moral Interpretation

- $(e^{\frac{t}{2}\triangle_{\mathcal{A}}}f(//_1))_a = (e^{\frac{t}{2}\triangle_K}f)_a(//_1^{\mathbb{C}})$

- So "restricting" to $\mathcal{A}$ and differentiating in t gives $\triangle_{\mathcal{A}}\left[f(//_1)\right] = (\triangle_K f)\,(//_1)$.

- Moreover,
$$\lim_{s\to\infty}\int_{\bar{\mathcal{A}}} f\left(//_1\left(A\right)\right)d\tilde{P}_s\left(A\right) = \int_K f\left(k\right)dk$$
showing Haar measure on K is the correct choice.

Corollary: Extended Hall's Transform

Let $\rho_s(dx) = Law(//_1)$ and $m_{s,t}(dg) = Law(//_1^{\mathbb{C}})$ so that

$$\rho_s(x) = \left(e^{s\Delta_K/2}\delta_e\right)(x) \text{ for } x \in K \quad \&$$

$$m_{s,t}(g) = \left(e^{A_{s,t}/2}\delta_e\right)(g) \text{ for } g \in K_{\mathbb{C}}.$$

Corollary 11 (A One Parameter family of Hall's Transforms)**.** *The map*

$$f \in L^2(K, \rho_s) \to \left(e^{t\Delta_K/2} f\right)_a \in \mathcal{H}L^2(K_{\mathbb{C}}, m_{s,t})$$

is unitary. Note that $m_{s,t}$ is the convolution heat kernel for $e^{A_{s,t}/2}$.

This theorem interpolates between the two previous versions of Hall's transform corresponding to $s = \infty$ and $s = \frac{t}{2}$.

Key Ingredients of the Proof 9

- Compute the action of the Segal-Bargmann transform on multiple Wiener integrals.
- Use the [Veretennikov & Krylov, 1976] formula twice to develop $f(//_1)$ and $F(//_1^{\mathbb{C}})$ into an infinite sum of multiple Wiener integrals (the Itô chaos expansion).
- Use these two items together to show $S_t\left[f(//_1)\right] = F(//_1^{\mathbb{C}})$.

Remark 12. See Dimock 1996, and Landsman and Wren $(\cong 1998)$ for other approaches to “canonical quantization” of YM_2.

Non - Closability of Δ_H when $d = \infty$

- $\|a\|_H^2 := \int_0^1 \dot{a}(t)^2 \, dt$ where $a(0) = 0$,
- Let μ be standard Wiener measure – so "informally"

$$d\mu(a) = \frac{1}{Z} \exp\left(-\frac{1}{2} \|a\|_H^2\right) \mathcal{D}a.$$

- Let $f(a) = 2\int_0^1 a_\theta da_\theta = a_1^2 - 1$ (Itô integral).
- On one hand,

$$\Delta_{H(\mathfrak{k})} f(a) = \sum_{h \in S_0} 2h_1^2 = 2.$$

- On the other hand, we have $f(a) = \lim_{|\mathcal{P}| \to 0} f_{\mathcal{P}}(a)$ where $f_{\mathcal{P}}(a)$ is the cylinder function

$$f_{\mathcal{P}}(a) = 2 \sum_{s_i \in \mathcal{P}} a_{s_i}(a_{s_{i+1}} - a_{s_i})$$

which are all Harmonic, i.e.

$$\Delta_{H(\mathfrak{k})} f_{\mathcal{P}}(a) = 0!$$

(Compare with the harmonic function

$$(x_1 + x_2 + \cdots + x_n)x_{n+1} \text{ on } \mathbb{R}^{n+1}.)$$

Therefore $\lim_{|\mathcal{P}| \to 0} f_{\mathcal{P}} = f$ while

$$0 = \lim_{|\mathcal{P}| \to 0} \Delta_{H(\mathfrak{k})} f_{\mathcal{P}}(a) \neq \Delta_{H(\mathfrak{k})} f = 2.$$

The Segal-Bargmann Transform

- $\mathcal{A} := \mathbb{R}^d$ and $\mathcal{A}_{\mathbb{C}} := \mathbb{C}^d$ with coordinate, $x \in \mathcal{A}$ and $z = x + iy \in \mathcal{A}_{\mathbb{C}}$.

- Let $\Delta_x = \sum_{\ell=1}^d \frac{\partial^2}{\partial x_\ell^2}$ and $\Delta_y = \sum_{\ell=1}^d \frac{\partial^2}{\partial y_\ell^2}$

- $A_{s,t} = (s - t/2)\, \partial_x^2 + \frac{t}{2}\partial_y^2$

- Let $r = 2(s - t/2)$, $x^2 = |x|^2$, $y^2 = |y|^2$,

$$\rho_s(x) = \left(e^{s\Delta/2}\delta_0\right)(x) = \left(\frac{1}{\sqrt{2\pi s}}\right)^d e^{-x^2/2s}$$

and

$$m_{s,t}(z) = \left(e^{A_{s,t}/2}\delta_0\right)(z) = \left(\frac{1}{\pi\sqrt{rt}}\right)^d e^{-x^2/r - y^2/t}.$$

Theorem 13 (Segal - Bargmann)**.** *For all* $s > t/2$, $z \in \mathbb{C}$ *and* $f \in L^2(\mathcal{A}, p_s(x)dx)$ *let*

$$S_t f := (\textit{Analytic Continuation}) \circ e^{t\Delta/2} f,$$

more explicitly,

$$(S_t f)(z) = \int_{\mathcal{A}} f(y) p_t(z-y) dy = \left(e^{t\Delta/2} f\right)_a (z).$$

Then

$$S_t : L^2(\mathcal{A}, p_s(x)dx) \to \mathcal{H}L^2(\mathcal{A}_{\mathbb{C}}, m_{s,t}(z)dz)$$

is a unitary map.

Sketch of the isometry proof

- Let $\partial_j := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$ and $\bar{\partial}_j := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$
- Let $f(x)$ be a polynomial in $x \in \mathcal{A}$,
- Let $f(z)$ be its analytic continuation to $z \in \mathcal{A}_{\mathbb{C}}$,
- Define $F_t(z) := \left(e^{-t\Delta_x/2} f\right)(z)$ so that $f = e^{-\frac{t}{2}\Delta_x} F_t = e^{-\frac{t}{2}\partial^2} F_t$.

- So

$$
\begin{aligned}
|f|^2 &= f \cdot \bar{f} = e^{-\frac{t}{2}\partial^2} F_t \cdot e^{-\frac{t}{2}\bar{\partial}^2} \bar{F}_t \\
&= e^{-\frac{t}{2}\partial^2} e^{-\frac{t}{2}\bar{\partial}^2} \left[F_t \cdot \bar{F}_t\right] = e^{-\frac{t}{2}\left(\partial^2+\bar{\partial}^2\right)} |F_t|^2 .
\end{aligned}
$$

- Next observe that

$$
\begin{aligned}
\left(\partial^2 + \bar{\partial}^2\right) &= \frac{1}{4}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)^2 + \frac{1}{4}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)^2 \\
&= \frac{1}{2}\left(\Delta_x - \Delta_y\right)
\end{aligned}
$$

- Therefore,

$$
\begin{aligned}
e^{\frac{s}{2}\Delta_x} |f|^2 &= e^{\frac{s}{2}\Delta_x} e^{-\frac{t}{2}\left(\partial^2+\bar{\partial}^2\right)} |F_t|^2 = e^{\frac{s}{2}\Delta_x - \frac{t}{4}(\Delta_x - \Delta_y)} |F_t|^2 \\
&= e^{\frac{1}{2}\left(\left(s-\frac{t}{2}\right)\Delta_x + \frac{t}{2}\Delta_y\right)} |F_t|^2 .
\end{aligned}
$$

- Conclusion,

$$
\begin{aligned}
\int_{\mathcal{A}} |f|^2 \, d\rho_s &= \left(e^{\frac{s}{2}\Delta_x} |f|^2\right)(0) = \left(e^{\frac{1}{2}\left(\left(s-\frac{t}{2}\right)\Delta_x + \frac{t}{2}\Delta_y\right)} |F_t|^2\right)(0) \\
&= \int_{\mathcal{A}_{\mathbb{C}}} \left|\left(e^{\frac{t}{2}\Delta_x} f\right)_a\right|^2 dm_{s,t}.
\end{aligned}
$$

Abstract Itô Chaos Expansion

For completeness, let me state (a bit informally) an abstract form of the Itô Chaos expansion.

Theorem 14 (Abstract Itô-Chaos Expansion)**.** *If μ is a Gaussian measure on a Banach space W, informally given by*

$$d\mu(x) = \frac{1}{Z} \exp\left(-\frac{1}{2} \|x\|_H^2\right) \mathcal{D}x,$$

where $H \subset W$, then every $f \in L^2(W, \mu)$ has an orthogonal direct sum decomposition as

$$f = \sum_{n=0}^{\infty} I_n(f) \tag{10}$$

where

$$I_n(f) := \frac{1}{n!} e^{-\frac{1}{2}\Delta_H} \left[x \to \left(\partial_x^n e^{\frac{1}{2}\Delta_H} f\right)(0)\right].$$

Proof Ideas

1. $f = e^{-\frac{1}{2}\Delta_H} e^{\frac{1}{2}\Delta_H} f,$

2. $e^{\frac{1}{2}\Delta_H} f$ is smooth and so
$$\left(e^{\frac{1}{2}\Delta_H} f\right)(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\partial_x^n e^{\frac{1}{2}\Delta_H} f\right)(0).$$

3. Combing items 1. and 2. explains Eq. (10).

4. By more elementary Taylor theorem arguments, on may show $\int_{\bar{H}} I_m(f) \overline{I_n(f)} d\mu = 0$ if $m \neq n$.

5. This is based on the identity,
$$\mathbb{E}\left[\left(e^{-\frac{1}{2}\Delta} p\right) \cdot \left(e^{-\frac{1}{2}\Delta} \bar{q}\right)\right] = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle (D^n p)(0), (D^n q)(0) \right\rangle_{(H^*)^{\otimes n}}.$$
which is valid for any polynomials p and q.

End

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