1. (a) Since $r^2 - 2r + 2 = 0$, we have $r = 1 \pm i$. Thus $y = e^t(C_1 \cos t + C_2 \sin t)$. Alternate forms: $C_1 e^t \sin(t + C_2)$ and $C_1 e^t \cos(t + C_2)$.

   (b) This is linear. Dividing by t, the integrating factor is $\exp(\int 2/t) = t^2$. Thus $(t^2 y)' = 4t^3$ and so $y = t^2 + C/t^2$. Since $y(1) = 3$, $y = t^2 + 2/t^2$.

   (c) Separating variables gives $(1 + y^2)dy = x^2\,dx$ and so $3y + y^3 = x^3 + C$.

   (d) Since $r^2 - 1 = 0$, the general solution to the homogeneous equation is $C_1 e^t + C_2 e^{-t}$. Hence we let $y = Cte^t$ and find C. Since $y'' = C(t + 2)e^t$, $y'' - y = 2Ce^t$ and so $C = 1$. Thus the general solution is $y = te^t + C_1 e^t + C_2 e^{-t}$.

   (e) (p.190 #13) Variation of parameters: Either use $y = u_1 y_1 + u_2 y_2$ or Theorem 3.7.1. For the latter, $W = y_1 y_2' - y_2 y_1' = (t^2)(-1/t^2) - (2t)(1/t) = -3$ and so a particular solution is

$$\begin{aligned} Y(t) &= -t^2 \int \frac{(1/t)(3 - /t^2)}{-3} dt + (1/t) \int \frac{t^2(3 - 1/t^2)}{-3} dt \\ &= \frac{t^2}{3} \int (3/t - 1/t^3) dt - \frac{1}{3t} \int (3t^2 - 1) dt \\ &= (t^2/3)(3 \ln t + 1/2t^2) - (1/3t)(t^3 - t) \\ &= t^2 \ln t + 1/2 - t^2/3. \end{aligned}$$

   Thus the general solution is $C_1 t^2 + C_2/t + t^2 \ln t + 1/2$.

   (f) (p.391) $\mathbf{x} = C_1 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

2. (a) The characteristic equation is $r^2 - 2r + 5 = 0$ and so the eigenvalues are $1 \pm 2i$.

   (b) As $t \to \infty$, $x_1(t)$ oscillates with increasing amplitude. (You need not be more specific.)

3. (a) (p.259 #5) Since

$$\begin{aligned} (1 - x)y'' &= (1 - x) \sum_{n \geq 0} n(n - 1)a_n x^{n-2} \\ &= \sum_{k \geq 0} (k + 2)(k + 1)a_{k+2} x^k - \sum_{k \geq 0} (k + 1)k a_{k+1} x^k, \end{aligned}$$

   the recursion is $(k+2)(k+1)a_{k+2} - (k+1)ka_{k+1} + a_k = 0$. (You need not specify where k starts in the recursion.) We could rearrange this (but need not for (a)):

$$a_{n+2} = \frac{n(n + 1)a_{n+1} - a_n}{(n + 1)(n + 2)}.$$

(b) By the initial conditions, $a_0 = 1$ and $a_1 = 0$. You can either use the recursion or differentiate the differential equation and evaluate at $x = 0$.
By the recursion

$$a_2 = \frac{0 - a_0}{2} = -1/2$$
$$a_3 = \frac{2a_2 - a_1}{6} = -1/6$$
$$a_4 = \frac{6a_3 - a_2}{12} = \frac{-1 + 1/2}{12} = -1/24$$

Using the differential equation, $y''(0) = -y(0)$, which is $2a_2 = -a_0$.
Hence $a_2 = -1/2$.
Differentiating, $(1 - x)y^{(3)} - y'' + y' = 0$ and so $y^{(3)}(0) = y''(0) - y'(0)$, which is $6a_3 = 2a_2 - a_1$. Hence $a_3 = -1/6$.
Differentiating again, $(1-x)y^{(4)} - 2y^{(3)} + y'' = 0$ and so $y^{(4)}(0) = 2y^{(3)}(0) - y''(0)$, which is $24a_4 = 12a_3 - 2a_2$. Hence $a_4 = -1/24$.

4. We have $F(s) = \dfrac{s}{(s-1)^2 + 2^2} = \dfrac{s-1}{(s-1)^2 + 2^2} + \dfrac{1}{(s-1)^2 + 2^2}$ and so $f(t) = e^t \cos 2t + (1/2)e^t \sin 2t$.

5. Taking the transform, $s^2Y(s) - sy(0) - y'(0) + 3Y(s) = G(s)$. Thus $Y(s) = \dfrac{s + 2 + G(s)}{3 + s^2}$.

6. Either write $g(t) = 1 - u_3(t)$ and use the table to get $G(s) = (1 - e^{-3s})/s$ or evaluate the transform directly:

$$G(s) = \int_0^3 e^{-st} dt = \left. \frac{-e^{-st}}{s} \right|_{t=0}^{3} = \frac{1 - e^{-3s}}{s}.$$