A vector (or scalar) function \( \mathbf{F}(\mathbf{R}) \) is called *homogeneous of degree* \( d \) if \( \mathbf{F}(t\mathbf{R}) = t^d \mathbf{F}(\mathbf{R}) \) for all vectors \( \mathbf{R} \) and real numbers \( t \) such that \( \mathbf{R} \) and \( t\mathbf{R} \) are in the domain of \( \mathbf{F} \). (If \( d = 0 \), we define \( 0^d = 1 \).)

**S1.** Suppose \( \mathbf{F}(\mathbf{R}) \) is defined in a domain that is star shaped with respect to the origin and is homogeneous of degree \( d \). Using the integral formulas from Sections 4.4 and 4.5 with \( \mathbf{R}_0 = 0 \), prove the following:

(a) If \( \mathbf{F} \) has a scalar potential, then \( \frac{1}{d+1} \mathbf{F}(\mathbf{R}) \cdot \mathbf{R} \) is a scalar potential for \( \mathbf{F} \) and it is homogeneous of degree \( d + 1 \).

(b) If \( \mathbf{F} \) has a vector potential, then \( \frac{1}{d+2} \mathbf{F}(\mathbf{R}) \times \mathbf{R} \) is a vector potential for \( \mathbf{F} \) and it is homogeneous of degree \( d + 1 \).

Remark: Since the star-shaped domain contains the origin, we may take \( t = 0 \) and \( \mathbf{R} \) any point in the domain to get \( \mathbf{F}(0\mathbf{R}) = 0^d \mathbf{F}(\mathbf{R}) \). It follows that \( d \geq 0 \).

**S2.** Show that the vector functions \( \mathbf{F} \) of Examples 4.9, 4.10 and 4.12 and the vector function in Exercise 4.5.2 are all homogeneous and compute their degrees.

**S3.** Using Exercises S1 and S2, derive vector (and, for Example 4.12, scalar) potentials for Examples 4.9, 4.10 and 4.12 and Exercise 4.5.2.

**S4.** For Example 4.12 and Exercise 4.5.2 the vector potentials \( \mathbf{G} \) obtained in the previous exercise do not agree with the result in the book. For instance, for Example 4.12 you should obtain \( \frac{1}{3}(-yz\mathbf{i} - xz\mathbf{j} + 2xy\mathbf{k}) \). Explain why both answers are correct.