A Matter of Proportion: Scaling in Sports and Elsewhere

by
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Some insects can jump remarkable distances. A flea can jump up about a foot — 100 times its height. At the rate, a person would be able to leap over tall buildings. Actually most people can't even jump up their own height and no animal about the size of a person can do much better. Why is this?

When the 1968 Summer Olympics were held in Mexico City, some people were concerned that reduced air resistance at that high altitude would be an unfair advantage leading to new records. Others thought that since less oxygen would be available at that altitude, times might increase rather than improve. How much difference should the change in altitude make?

The formula for the period of a pendulum is rather simple. Why is this?

1 High Jumpers: People Versus Fleas

We want to understand why humans are such weaklings compared to fleas in jumping up.

The key to studying jumping height is to look at energy. Contraction of muscles provides energy, so the amount of energy available to an organism in a short time should be roughly proportional to the mass of its muscles. This mass should be roughly proportional to the size of the animal. Thus we expect

$$\text{(available energy)} \approx \text{C(body mass)},$$

for some constant C. The energy needed to raise a mass m a distance h equals mgh, where g is the acceleration due to gravity.[1] If all the available muscle energy were used to lift an organism a distance h, we would have $Cm \approx mgh$, where m is the mass of the organism. Thus

$$h \approx C/g, \tag{1}$$

regardless of the organism, provided it was doing an efficient job of converting muscle energy into body lifting. In other words, height depends only on the acceleration due to gravity — it does not depend on size.

Since most people can jump over a foot (the height a flea can jump), it's the fleas that are weaklings, not people!

Actually, that may not be true. We neglected two important things:

(a) "center of mass", which is important for people, and

(b) air resistance (drag), which is important for fleas.

Until we understand them, we don't know whether the flea or the human should get the gold medal.

Center of Mass The center of mass is a point in an object where we can imagine all the mass to be located for the purposes of studying the object's motion. It concerns us here because the height h should be thought of as the change in height from the starting center of mass to the highest point reached by the organism. Compared to the size of its jump, a flea is practically a dot, so there isn't much correction needed. An average standing person has a center of gravity that is about two feet off the ground. Suppose a good jumper clears a bar that is seven feet high. If he clears the bar horizontally, he raises his center of gravity to a point a bit over seven feet above the ground. He can do better than this by going over the bar in a bent fashion, he can keep his center of gravity below seven feet![2] Thus, while the athlete was credited with a seven-foot jump, his center of gravity may have made a jump of less that five feet — still better than the flea.

[1] For now, you either have to know this from physics or accept it on faith. We'll derive it later.

[2] We won't explain this here. You can find an explanation in the solution to Problem 73 in Jargocki's book [2].

Drag We now turn to the effects of drag. The explanation is a bit technical. When we say "x is proportional to y" in the following discussion, we mean this to be only roughly true. Thus we mean that there is a constant C such that $x \approx Cy$.

To begin with we need to take on faith a little bit of physics.

- First, a object with mass m and speed v has kinetic energy $mv^2/2$, i.e.

$$\text{(kinetic energy)} \;=\; C_1 m v^2. \tag{2}$$

  This is the energy an organism obtains from its muscles.

- Second, for similarly shaped objects, drag force is proportionaly to Sv^2 where S is surface area and v is velocity and so the the energy lost to drag in rising to a height h is proportional to Sv^2h where v is the starting velocity. Let's restate that:

$$\text{(energy lost to drag)} \;\approx\; C_2 S v^2 h. \tag{3}$$

Let L be a "typical distance" on an organism; for example, its diameter when it rolls up as close to a ball as it can. It's volume is proportional to L^3 and its surface area is proportional to L^2. Since mass is proportional to volume, the organism's mass is proportional to L^3. Thus

$$S \;\approx\; C_3 L^2 \;=\; C_3 L^3/L \;\approx\; C_4 m/L.$$

Using this in (3) and then using (2), we have

$$\text{(energy lost to drag)} \;\approx\; C_5 m v^2 h/L \;\approx\; \text{(kinetic energy)} C_6 h/L. \tag{4}$$

By conservation of energy,

$$\text{(kinetic energy)} \;=\; \text{(energy lost to drag)} + mgh. \tag{5}$$

Remembering that the kinetic energy provided by muscle is proportional to mass and combining (5) with (4), we have

$$C_7 m \;\approx\; (C_7 m) C_6 h/L + mgh.$$

Solving for h:

$$h \;\approx\; \frac{C_7}{g + C_6 C_7/L} \;=\; \frac{1}{g/C_7 + C_6/L}.$$

Note that h gets smaller as L gets smaller since the denominator gets larger as L gets smaller. Thus we don't expect fleas to jump as high as people. Since we don't know the values of the constants C_6 and C_7, we can't say exactly how things change. Hence we can't decide if a flea or a human is a better jumper.

Pole Vaulters We've learned that high jumpers use their muscles to provide energy to raise their center of mass to a certain height. The higher they want to raise it, the more energy they need. A pole vaulter can go nearly three times as high as a high jumper. Where does he get all the extra energy?

Instead of jumping over the bar, he runs over it! He uses the pole to change the direction of his running kinetic energy from a horizontal direction to a vertical direction. (Technically, it's his momentum that changes direction since energy does not have direction.) First, his kinetic energy is used to bend the pole, which stores energy in the pole. He also uses the pole to greatly reduce his forward momentum. When the pole straightens again, it imparts energy and momentum to the vaulter. By properly positioning the pole, the momentum will be imparted in a vertical direction. A person jumping from a standing position only has the energy from flexing his leg muscles once. A pole vaulter has the energy from flexing them several times as he builds up his running speed.

2 Mexico City Advantage: Fact or Fiction?

Let's look at runners.

- How much does drag slow down a runner? Without knowing more about physics, we can't say. All we know is that drag increases with speed, so it should slow down sprinters more than it slows down long distance runners. Thus, if the reduced drag in Mexico City helped, it would help sprinters the most.

- How much does the thinner air slow down runners from lack of oxygen? One can build up oxygen by deep breathing before running and one can build up a bit of an "oxygen debt." Thus oxygen should not be a problem for short distance runners, but may be a problem for long distance runners.

Thus, although we don't know if either drag or lack of oxygen matter, they both act in the same way: any benifits will be most noticeable over short distances and any handicaps will be most noticeable over long distances. Here are the winning times for men from some olympics.

place year	Rome 1960	Tokyo 1964	Mexico 1968	Munich 1972	Montreal 1976
100m	10.2	10.0	9.95	10.14	10.06
200m	20.5	20.3	19.83	20.00	20.23
5000m	13:43.4	13:48.8	14:05.0	13:26.4	13:24.8
10000m	28:32.2	28:24.4	29:27.4	27:38.4	27:40.4
marathon	2:15:16.2	2:12:11.2	2:20:26.4	2:12:19.8	2:09:55.0

It appears that reduced air resistance *might* have helped on the shorter races, but it could be random variation or a psychological effect due to expecting better performance. A decision on that will depend on what the physicists say. There is stronger evidence that oxygen problems impaired the runners in the longer races. It seems fairly safe to guess that this was a problem.

The air is not the only difference. Another problem is gravity. We see in (1) that height increases as gravitational acceleration decreases: Decreasing g by p percent should increase height reached by about p percent. The value of g varies from place to place. This variation is due mainly to altitude and latitude. (Because the earth tends to bulge out nearer the equator, the latitude effect tends to be cancelled out.) For example, on page 113 Jargocki [2] says that $g = 978.44$ m/sec^2 in Mexico City and 981.50 in Oslo. This is a difference of 0.3%. Thus a high jumper or a pole vaulter should be able to raise his center of gravity about 0.3% more in Mexico City than in Oslo. For a high jumper, this is about

$$0.3\% \times 5 \text{ feet} \quad \approx \quad 0.2 \, inches,$$

not very much, but it could make a difference. Pole vaulters go higher and so we have a bigger difference for them:

$$0.3\% \times 17 \text{ feet} \quad \approx \quad 0.6 \, inches.$$

In conclusion, it appears that the site of the 1968 Summer Olympics did not have a major effect on the outcome except that oxygen deprivation may have prevented new records from being set in the longer races.

3 Dimensional Analysis

For simplicity, we'll work limit our attention to dynamics — the study of moving bodies — and our universe will be Newtonian.

The basic units of measurement in dynamics are mass, length and time, which we denote by M, L and T. Since speed is obtained by dividing distance by time, we say that its units are L/T. Similarly,

- the units of acceleration are L/T^2,
- the units of force, which is mass times acceleration, are ML/T^2
- the units of energy, which is force times distance, are ML^2/T^2 and
- the units of angles, which are arc length divided by radius are $L/L = 1$.

We say that angles are *dimensionless*. Similarly, Fh/mv^2, which has units $(ML/T^2)L/M(L/T)^2 = 1$ is dimensionless. The basic concept behind dimensional analysis is

> Every physical formula can be rewritten as the statement that some dimensionless product p is a function of dimensionless products, none of which contain powers of p. (6)

"Product" here allows division as well as multiplication. If all dimensionless products are powers of p, the function has no variables and so is constant.

What good is (6)? Let's do some examples.

Kinetic Energy Due to its motion, an object possesses energy, called kinetic energy. Let E_K be the kinetic energy of an object. What could possibly affect the value of E_K? After a little thought, it will probably seem clear to you that only an object's mass m and speed v should matter. Every dimensionless product that can contain only E_K, m and v must be a power of E_K/mv^2. Why? Recall that m and v have units M and L/T. The units of E_K, since it is energy, are ML^2/T^2. The only way we can combine ML^2/T^2, M and L/T to get something dimensionless is

$$\frac{(ML^2/T^2)^a}{M^a(L/T)^a} = \left(\frac{ML^2/T^2}{M(L/T)}\right)^a.$$

Thus, our law must have the form $E_K/mv^2 = C$ for some constant C. Solving for E_K: $E_K = Cmv^2$. Dimensional analysis does not let us determine the value of constants, so we cannot find C. It turns out that $C = 1/2$.

Potential Energy An object gains potential energy when it is raised in a gravitational field. Suppose the gravitational field is constant and let g be the acceleration due to gravity. The potential energy E_P could depend on g, the mass m of the object, the height h it is raised and the time t it takes to raise it.

When an object falls, its potential energy is converted into kinetic energy. By observation, how fast an object falls does *not* depend on how long it took to raise it up. Thus the potential energy does not depend on t.

We've reduced our list of quantities to E_P, g, m and h, which have dimensions ML^2/T^2, L/T^2, M and L. We leave it to you to show that all dimensionless products are powers of E_P/mgh. Thus $E_P/mgh = C$ and so $E_P = Cmgh$ for some constant C. In fact, $C = 1$.

The Period of a Pendulum To simplify matters, we will neglect things like air resistance and friction that gradually slow a pendulum down. Also, we'll assume that the pendulum is a weight at the end of a string, where the string is so light that its mass can be ignored and the linear dimensions of the weight can be ingored because they are much smaller than the length of the string. We are interested in finding the period of the pendulum; that is, the length of time t that it takes for one complete swing. What aspects of the situation matter? Here are the possibilities we came up with:

- t, the period that we want a formuls for, which has dimension T,
- θ, the maximum angle of the swing, which is dimensionless,
- g, the acceleration due to gravity, which has dimension L/T^2,

- ℓ, the length of the pendulum's string, which has dimension L, and
- m, the mass of the weight, which has dimension M.

One dimensionless product is θ. Another is $t^2 g/\ell$. There are none that involve m and all dimensionless products have the form $\theta^a (t^2 g/\ell)^b$. Thus we can write $t^2 g/\ell = f(\theta)$ for some function f. Solving for t and letting $F(\theta 0 = \sqrt{f(\theta)}$, we have

$$\text{(period of a pendulum)} \;=\; F(\theta)\sqrt{\ell/g},$$

for some function θ that cannot be determined by dimensional analysis. It turns out that the function F is nearly constant for small angles θ.

Since the examples involve unknown constants and an unknown function $F(\theta)$, what use are they? If we lack the ability to calculate the constants or the function $F(\theta)$, we could determine them experimentally. This reduces the work considerably. For example, without that reduction, we would only know that $t = H(\theta, m, g, \ell)$ for some function H of four variables. Because of all the variables, the difficulty of varying g and the need to explore very small and very large values of t, it would be much harder to construct a table of this function experimentally than to construct a table of $F(\theta)$. For $F(\theta)$, we can simply keep g at whatever value we happen to find in the laboratory and choose a value for ℓ that puts t in a convenient range — neither too large nor too small.

References

[1] Edward A. Bender, *An Introduction to Mathematical Modeling*, J. Wiley & Sons (1978). Reprinted by Dover. Chapter 2 contains a variety of material on arguments from scale.

[2] Christopher P. Jargocki, *Science Brain-Twisters, Paradoxes, and Fallacies*, Charles Schribner's Sons (1976).