“And the winner is . . .”

by
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1 The Better of Two Teams

Let’s look at the simplest case first: two competitors, say the Comets and the Meteors. If they play each other enough, the better one will win more games and so we’ll know which is better. To illustrate, suppose the Comets would win a game 55% of the time.

- If they play once, then there is a 45% chance the Meteors would win, a minor upset since the Comets are slightly better.
- If they play twice, then there is about a 50% chance of a tie, which is not good.
- If they play three times, then there is about a 42.5% chance that the Meteors have an upset win.

At this rate, how many games would be needed until the Comets would had only a 20% chance of scoring an upset? The answer is 71 — a lot of games to play. Here is a table of how many games it takes to reduce the probability of upset below 30% 20% and 10% for various probabilities of the underdog winning a single game. The bottom line will be explained after the table.

	0.45	0.40	0.35	0.30
0.30	27	7	3	1
0.20	71	17	7	5
0.10	163	41	17	9
upset	0.39	0.29	0.20	0.13

From this we can see that, if two teams are fairly closly matched, upsets are common. How common are upsets in the World Series? In this case, a team must win four out of seven games. The bottom line gives the probability of an upset when seven games are played.

2 Three Teams

What happens with three teams, the Comets, the Meteors and the Asteroids when, from game to game, a team may vary in its ability? The ability doesn't depend on whom they're playing, it's just that teams have good days and bad days. Suppose we somehow know what these abilities are and we want to decide which team is best.

Let's use the numbers 1 to 6 to indicate ability. In any particular game, the team with the higher number wins. Here's the strength information:

- Asteroids: always 3. They're average but steady.
- Comets: 1/3 of the time at 6 and 2/3 of the time at 2. Sometimes they're tops but mostly they're somewhat below average.
- Meteors: 2/3 of the time at 4 and 1/3 of the time at 1. Mostly they're somewhat above average, but sometimes they're lousy.

Which team do you think is the best? Why? Maybe it's the Comets who, at their best (a 6), can beat the other teams. Maybe it's the Meteors who more often than not are above average. Maybe it's the average but steady Asteroids.

This isn't getting anywhere, so maybe we should look at the average ability of each team. The average for the Asteroids is clearly 3. Since the Comets are at a 6 for 1/3 of the time and at a 2 for 2/3 of the time, their average is

$$\frac{1}{3} \times 6 \; + \; \frac{2}{3} \times 2 \;=\; \frac{10}{3}.$$

Similarly the average for the Meteors is 3. Thus

the Comets are the winners!

This is a mistake. Can you see why before reading on?

The numbers 1–6 are only rankings of abilities. Thus we could change the Meteors' 4 to a 5 and the Asteroids' 3 to a 4 without changing the outcome of games between the teams. Now the averages are

Asteroids: 4 Comets: 10/3 Meteors: 11/3,

and so the Comets are the worst team now and the Asteroids are best.

What should we do? We could do the "obvious" thing — look at how the teams would do against one another. Since the Asteroids are always a 3, they beat the Comets 2/3 of the time and lose to the Meteors 2/3 of the time. The Comets versus Meteors is a more complicated situation.[1] The games the Comets play can be divided into 3 equal groups: In one group their ability is 6 and in the other two their ability is 2. We can do the same thing for the Meteors and then pair up the possibilities. Here's a picture that shows what happens

	Comets = 6	Comets = 2	Comets = 2
Meteors = 4	4 vs. 6	4 vs. 2	4 vs. 2
Meteors = 4	4 vs. 6	4 vs. 2	4 vs. 2
Meteors = 1	1 vs. 6	1 vs. 2	1 vs. 2

Thus, the Meteors win in the four cases in the upper right and the Comets win in the other five cases. In summary

- Asteroids beat Comets 2/3 of the time.
- Comets beat Meteors 5/9 of the time.
- Meteors beat Asteroids 2/3 of the time.

[1] If you know some probability, you can probably work out the answer, but we want to do it without that since this is a Freshman Seminar.

Thus, the Asteroids seem better that the Comets who seem better than the Meteors who seem better than the Asteroids.[2]

Of course, this is all about sports, so we can resolve the problem rather easily — look at what happens when a lot of games are played. Each team plays half its games against each of the others. When the Asteroids play the Comets, they win 2/3 of their games and when they play the Meteors, they win 1/3 of their games. So half the time they win 2/3 and half the time 1/3 giving an average of

$$\frac{1}{2} \times \frac{2}{3} + \frac{1}{2} \times \frac{1}{3} = \frac{1}{2}.$$

Similarly, the Comets win $(1/2) \times (1/3) + (1/2) \times (5/9) = 4/9$, and the Meteors win 5/9 of their games. Thus the Meteors are the best team, at least by total number of wins.

3 Sportswriters Vote for the Best

Before the season started, we asked twenty-one sportswriters to rank the Asteroids, Comets and Meteors so that we could pick the best team. Here's the results of the ranking, with A, C and M used to denote the teams.

votes	8	7	3	2	1
1st	A	C	M	M	A
2nd	M	M	A	C	C
3rd	C	A	C	A	M

(1)

Looking at the first place votes, it's clear that the Asteroids are the winners with nine votes and the Meteors are the last choic with only six votes.

Maybe its not as clear as it looks. Using the information in the table, we

[2]You can use this in a game with a friend. Take three dice but change the numbers on them: On one put all 3's, on another, two 6's and four 2's, and on the third, two 1's and four 5's. Here's how you play the game. Your friend gets to choose a die, then you choose one of the remaining ones, both of you roll and the winner is the one with the higher roll. If you friend chooses an "Asteroid" or "Comet" die, you can win about 2/3 of the time. If he chooses a "Meteor" die, you can still win 5/9 of the time. You can make some money, but you risk losing a friend.

can ask what the sportswriters would say in a two team runoff:[3]

Meteors beat Asteroids	12 to 9
Meteors beat Comets	13 to 8
Asteroids beat Comets	12 to 9

Anyone seeing just these results would declare the Meteors the winners. There are at least two questions that may occur to you:

- How did it happen that the Meteors moved from last choice to first?
- What is the "correct" thing to do?

The first question is an easy one: It happened because the Meteors, were often the first or second choice.

The second question can be said to not have an answer. A variety of voting schemes have been proposed, but all have shortcomings. What does "shortcoming" mean? It means the voting system fails to satisfy some reasonable condition. This may not be due to a lack of creativity in designing a system. To illustrate, A system where teams are ranked as in (1) is called a ranking system of voting. The goal is to combine the votes to produce a combined ranking, with ties allowed. Here are some reasonable axioms:

A1. (Universality) The system must produce a combined ranking regardless of how individuals ranked the teams. Also, it must work with any number of voters and any number of teams.

A2. (Unanimity) If all voters consider team X to be at least as good as team Y, then this is true in the combined ranking. Furthermore, if at least one voter said that team X was better than Y and the remaining voters considered them equal, then X is better than Y in the combined ranking.

A3. (Improvement) Suppose that team X is preferred to Y in the final ranking. If a revote is taken and, everyone who preferred team X to Y still does so, then X is still preferred to Y in the combined ranking.

[3] For example, to get the first row, add up the columns in the table where the Meteors were ranked above the Asteroids ($7 + 3 + 2 = 12$) and those where the Asteroids were ranked above the Meteors ($8 + 1 = 9$).

In 1950, Kenneth Arrow proved that the only way to satisfy all three axioms was to designate one voter as dictator and use his (or her) ranking for the combined ranking. This is known as the *Arrow Impossibility Theorem*. One might object to these axioms, especially number 3. There are a variety of results that show that a combined ranking is still impossible.

Problems like these are of practical importance. It gets even more complicated when there are several candidates for several positions; for example, there may be seven candidates for three positions on a board. Voters may be asked to rank the candidates, vote for three candidates, or divide three votes among the candidates. (The last two options are different: I might give all three of my votes to one candidate.) Suppose I would like to see Frances elected; however, I know that she is popular and so will probably win. Thus I might decide to use my voting power elsewhere. Of course, if enough other people are as clever as I, Frances might not get elected to the board after all!

If you'd like to learn more about voting, see [5, 6].

4 Dividing Resources

Somewhat related to voting is the problem of fair division of resources. Such problems are often referred to as *cake cutting*. Two books on the subject are [2, 4].

The U.S. Constitution mandates a census every ten years followed by a reapportionment of House of Representatives based on population, except that each state must have at least one representative. Also, at the present time, there must be a total of 435 representatives. Since a state cannot have a fractional number of representatives, such as $4\frac{1}{3}$, rounding must be done. While this might seem a simple problem, its not. For more on this, you can look at Balinski and Young's article [1], which also contains a variety of references.

5 Tournaments

To select the best team, we need to have the teams play each other and then make a decision. One of the simplest is what is known as a *round-robin* *tournament*. In such a tournament a number k is chosen and each team plays every other team exactly k times. We'll look at the case $k = 1$, which is the traditional round-robin tournament. First we need to design a schedule, then we need to choose a winner.

A simple schedule is to have every game on a different day. Is this practical? Supppose there are N teams. We will show that there are $N(N-1)/2$ games to be played. If the league has 10 teams, this means there will be 45 games. If we want each game on Saturday because it's Little League, we'll have to schedule multiple games on the same day. We'll look at this in a minute, but first: Why are there $N(N-1)/2$ games?

Call the teams $1, \ldots, N$ and list all pairs of teams with the larger number in the pair first:

$$(2,1) \quad (3,1)\;(3,2) \quad (4,1)\;(4,2)\;(4,3)\;\ldots\;(N,1)\ldots(N,N-1). \tag{2}$$

Note that one pair begins with 2, two pairs begin with 3, ... and $N-1$ pairs begin with N. Hence we want to add up the first $N-1$ integers to get the number of games needed. Call this sum S. Look at the following additions by rows and by columns.

$$\begin{array}{ccccccccccccc}
1 & + & 2 & + & 3 & + & \cdots & + & N-2 & + & N-1 & = & S \\
\underline{N-1} & + & \underline{N-2} & + & \underline{N-3} & + & \cdots & + & \underline{\quad 2 \quad} & + & \underline{\quad 1 \quad} & = & \underline{S} \\
N & + & N & + & N & + & \cdots & + & N & + & N & = & X
\end{array}$$

where $X = 2S$ by adding up the rightmost column and $X = N(N-1)$ by adding up the N's in the bottom row. Thus $2S = N(N-1)$ and so $S = N(N-1)/2$.

How many games can we schedule at the same time (assuming playing space is available)? If N is even, we can construct $N/2$ pairs of teams and so schedule $N/2$ games at a time. Thus, we need *at least* $\frac{N(N-1)/2}{N/2} = N-1$ times to schedule all games. When N is odd, we can only schedule $(N-1)/2$ pairs of teams at one time and so need *at least* $\frac{N(N-1)/2}{(N-1)/2} = N$ times. Can we schedule the tournament in this few time slots? Yes, as we'll show.

Suppose we know how to schedule a tournament for N teams at $N-1$ times whenever N is even. Then doing an odd number of teams is easy:
Suppose we have M teams and M is odd. Create team $M+1$, called "NoGame," for a total of $M+1$ teams. Since M is odd, $M+1$ is even. Call this

even number N. By the assumption at the start of this paragraph, we know how to schedule a tournament for these N teams at $N-1 = (M+1)-1 = M$ times. This is our schedule for the original M teams, with the understanding that when a team plays NoGame, it means that the team has no game at that time.

We still have to show how to schedule N teams at $N-1$ times when N is even. Before continuing, you should experiment with finding a way to do this. (There is more than one way to create such a schedule.)

We'll present a scheduling algorithm found by Reisz in 1859 and discussed in Chapter 14 of Moon's book [3]. For all pairs of integers i and j between 1 and $N-1$, let

$$r = \begin{cases} i+j, & \text{if } \ i+j < N \\ i+j-(N-1), & \text{if } \ i+j \geq N. \end{cases} \tag{3}$$

You should convince yourself that r is always between 1 and $N-1$. Now,

$$\begin{aligned} &\text{If } i \neq j, \text{ schedule } (i,j) \text{ on day } r. \\ &\text{If } i = j, \text{ schedule } (i,N) \text{ on day } r. \end{aligned} \tag{4}$$

We must show that every pair is on the schedule and there are no conflicts. To do this, it will be easier if team N is not treated differently. Hence we'll write (i,j) for day r, even if $i = j$. When it comes to actually playing, we'll interpret (i,i) and (i,N). It should be clear that every pair (i,j) with $i,j < N$ appears on the schedule exactly once, namely on the day r calculated by (3). What does "no conflicts" mean? It means that, on each day, each of the teams $1,2\ldots$, and $(N-1)$ appears in at most one game.

Suppose we have a conflict. This means that, for some i,j,k we scheduled (i,j) and (i,k) on the same day, say r. We can assume $1 \leq j < k$¡N. Note that $k-j < N-1$. Then either

$$\begin{array}{lrcccl} & r & = & i+j & = & i+k \\ \text{or} & r & = & i+j-(N-1) & = & i+j-(N-1) \\ \text{or} & r & & i+j & = & i+j-(N-1) \end{array}$$

If $i+j = i+k$, then $j = k$, a contradiction. If $i+j-(N-1) = i+k-(N-1)$, then $j = k$, a contradiction. If $i + j = i + k - (N - 1)$, then cancelling i and rearranging gives us $k - j = N - 1$. This is also a contradiction. This completes the proof.

What about choosing a winner? Again, this can be difficult. Some ideas about this and many other aspects of tournaments are discussed by Moon [3].

References

[1] Michel L. Balinski and Hobart P. Young, The apportionment of representation, pp. 1–27 in *Fair Allocation*, Hobart P. Young, (ed.), Proceedings of Symposia in Applied Mathematics, Volume 33, American Mathematical Society (1985).

[2] Steven J. Brams and Alan D. Tayler, *Fair Division: From Cake-Cutting to Dispute Resolution*, Cambridge Univ. Press (1996). For an idea of what's in the book, you can read the several page review by William F. Lucas in *The American Mathematical Monthly* **105** (1998) 877–881.

[3] John W. Moon, *Topics in Tournaments*, Holt, Reinhart & Winston, 1968. Parts of this book will be too difficult for you; however, you can read some parts of it.

[4] Jack Robertson and William Webb, *Cake-Cutting Algorithms: Be Fair If You Can*, A.K. Peters (1998). This book contains exercises. Exercises are useful for developing a better understanding of a subject.

[5] Donald G. Saari, *Chaotic Elections! A Mathematician Looks at Voting*, American Mathematical Society (2001).

[6] Alan D. Taylor, *Mathematics and Politics: Voting, Power and Proof*, Spring-Verlag (1995). For information about this book and references to several others, see the review by Samuel Merrill, III in *The American Mathematical Monthly* **104** (1997) 82–85.