1. (a) An integrating factor is \(e^{t^2/2}\) and so \(e^{t^2/2}y = \int t^3e^{t^2/2}dt\). The integral can be done using integration by parts with \(u = t\) or it can be done by substitution with \(x = t^2/2\) and then integration by parts. The result is \((t^2 - 2)e^{t^2/2} + C\) and so \(y = t^2 - 2 + Ce^{-t^2/2}\). Using the initial condition, \(0 = -2 + C\) and so \(C = 2\).

(b) Separate variables: \(-y dy = -t^{-2}dt\). Thus \(-e^{-y} = t^{-1} + C\).

(c) Set \(y = e^{rt}\) to obtain \(r^2 - 2r + 2 = 0\) and so \(r = 1 \pm \sqrt{-1}\). Hence the general solution is \(y = C_1e^{(1+i)t} + C_2e^{(1-i)t}\), which can be written \(y = e^t(D_1 \sin t + D_2 \cos t)\).

(d) The homogeneous equation \(y'' - 4y' + y = 0\) gives \(r^2 - 4r + 4 = 0\) and so \(r = -2\) is a double root. Thus the general solution to the homogeneous equation is \(y = (C_1 + C_2t)e^{2t}\). One can use variation of parameters or undetermined coefficients to find a particular solution. Using the latter, we set \(y = Ce^t\) and obtain \(y'' - 4y' + 4y = Ce^t\) and so \(C = 2\). Thus the general solution is \(y = (C_1 + C_2t)e^{2t} + 2e^t\).

(e) In matrix notation, \(x' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} x\). The determinant of \(\begin{pmatrix} 2 - r & -1 \\ 3 & -2 - r \end{pmatrix}\) is \(r^2 - 1\) and so \(r = \pm 1\). Using \(x = e^t \mathbf{c}\) we obtain

\[
e^t \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^t \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^t \begin{pmatrix} 2c_1 - c_2 \\ 3c_1 - 2c_2 \end{pmatrix},
\]

which has a solution \(e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}\). With \(x = e^{-t} \mathbf{c}\) we obtain

\[
-e^{-t} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^{-t} \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^{-t} \begin{pmatrix} 2c_1 - c_2 \\ 3c_1 - 2c_2 \end{pmatrix},
\]

which has a solution \(e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}\). Thus the general solution is

\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = d_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + d_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

2. (a) Let \(y = \sum a_n x^n\). Then

\[
y'' - xy' - y = \sum (n + 1)(n + 2)a_{n+2}x^n - \sum na_n x^n - \sum a_n x^n
\]

\[
= \sum (n + 1)((n + 2)a_{n+2} - a_n)x^n,
\]

and so \(a_{n+2} = \frac{a_n}{n+2}\) is the recursion.

(b) We are given \(a_0 = 1\) and \(a_1 = 2\). It follows that

\[
a_2 = \frac{a_0}{2} = \frac{1}{2}, \quad a_3 = \frac{a_1}{3} = \frac{2}{3}, \quad a_4 = \frac{a_2}{4} = \frac{1}{8}, \quad a_5 = \frac{a_3}{5} = \frac{2}{15}.
\]
3. \[
\frac{d^2 \theta}{dt^2} = \frac{d\omega}{dt} = \frac{d\omega}{d\theta} \frac{d\theta}{dt} = \frac{d\omega}{d\theta} \omega.
\]

(b) Separate variables and integrate to get \(\omega^2/2 = K \cos \theta + C\).

4. \(L[y''(t)] = s^2 Y(s) - s - 2, \quad L[y(t)] = Y(s)\) and

\[
L[g(t)] = \int_0^\infty g(t)e^{-st}dt = \int_0^1 (1-t)e^{-st}dt
\]

\[
= -\frac{(1-t)e^{-st}}{s}\bigg|_0^1 - \frac{1}{s} \int_0^1 e^{-st}dt = \frac{1}{s} - \frac{1-e^{-s}}{s^2}.
\]

Thus

\[
Y(s) = \frac{s^3 + 2s^2 + s - 1 + e^{-s}}{s^2(s^2 - 1)}.
\]