Another Way to Slice the Torus

In Exercise 6.2.61 (Stewart, 5th editon)[1] we are asked to find the volume of a torus. Since this is in the section on the washer method for volumes of revolution, we are expected to use washers. This leads to a messy integral. By slicing differently, we can avoid integrals and get a *much* more general result.

Definition A torus is constructed as follows. A disk of radius r is moved along a circle of radius $R > r$ in so that the center of the disk moves along the circumference of the circle and the disk is always perpendicular to the circumference at the point of intersection.

1. After discussing computation of volume by washers in Section 6.2, we find the torus problem (#61). As Stewart points out, the resulting washer integral leads to a somewhat messy integral which can be seen to be the area of a disk of radius r.
2. Instead of washers, use a "polar"[2] approach by taking wedges perpendicular to the plane of the circle. Let $\Delta\theta$ be the angle of the wedge. The result is almost a cylinder with a circular base except
   (a) The sides are not perfectly straight, but this can be ignored in the limit as $\Delta\theta \to 0$. (Not proved here, just asserted.)
   (b) The bases are not parallel: If we stand the "cylinder" on one base, the top base makes an angle $\Delta\theta$ with the horizontal. The shortest and longest "vertical" sides are $(R - r)\Delta\theta$ and $(R + r)\Delta\theta$. A horizontal plane at height $R\Delta\theta$ slices off a semicircular wedge on the high sideEœ which can be flipped to over to fill the low side. The result is a cylinder of height R.

   By stacking up all these cylinders, we obtain a cylinder whose height is the circumference of the circle. Hence the volume is $(2\pi R)(\pi r^2)$.
3. While a circle is essential for the circular washer argument in Item 1, it is not important in Item 2. Hence the circle in the definition of a torus can be replaced by any curve in 3-space, provided that the disk never overlaps itself as it moves along the curve. This means two points on separate regions of the curve are always at least $2r$ apart. Locally, one is led to the notion of "radius of curvature,"[3] which must be at least r.
4. The argument in Item 2 works because the symmetry of the circle guarantees that the semicircular wedge that is cut out will fit into the low part of the cylinder to level it off. Of course, all that matters is that the volumes be equal. An analysis of this shows that any figure can be used in place of the disk provided its center of gravity moves along the curve. We've generalized quite a bit from the torus problem!

[1] It is 6.2.59 in 4th edition.
[2] The usage of the term polar should be clearer after Section 10.4.
[3] This is discussed in a later course in the calculus sequence.