

We want to express geometric properties or operations on varieties in terms of their functions, i.e. their coordinate rings.

Classically: $X \xrightarrow{\varphi} Y$ cont. map of top. spaces.

given a cont. function $f: Y \rightarrow S$, we obtain a cont function $X \rightarrow S$ by composition:

$$\varphi^*(f) := f \circ \varphi: X \xrightarrow{\quad} Y \xrightarrow{\quad} S$$

this is the pull-back of functions.

For affine spaces A^m, A^n , the rings of functions

are $k[x_1, \dots, x_m]$ and $k[y_1, \dots, y_n]$.

Given a polynomial function $P \in k[y_1, \dots, y_n]$
 $P: A^n \rightarrow k$ function.

Else pull-back of P is the composition:

$$A^m \xrightarrow{\varphi} A^n \xrightarrow{P} k \quad \varphi \text{ cont.}$$

$$\varphi^* P = P \circ \varphi \in k[x_1, \dots, x_m]$$

So $\varphi^* : k[y_1, \dots, y_n] \longrightarrow k[x_1, \dots, x_m]$
ring hom.

condition on $\varphi : \varphi^*$ has to send pol. to pol.

$$\varphi : k^m \longrightarrow k^n \quad (\text{if } k \text{ is alg. closed})$$
$$(a_1, \dots, a_m) \longmapsto (b_1, \dots, b_n)$$

If $P(b_1, \dots, b_n) = 0$, then $P \circ \varphi(a_1, \dots, a_m) = 0$
" $(\varphi^* P)(a_1, \dots, a_m) = 0$

$$\text{So } \varphi^*(I(b_1, \dots, b_n)) \subseteq I(a_1, \dots, a_m)$$

Given a map of algebras $k[y_1, \dots, y_n] \xrightarrow{f} k[x_1, \dots, x_n]$
we can try to define a map $\varphi: \mathbb{A}^n \rightarrow \mathbb{A}^n$ s.t. $\varphi^* = f$.

Given a maximal ideal \mathfrak{m} in $k[x_1, \dots, x_n]$, we can
try to define $\varphi(\mathfrak{m}) := f^{-1}(\mathfrak{m}) \subset k[y_1, \dots, y_n]$.

However: the inverse image of a maximal ideal is
not always maximal! It is, however, prime.

The inverse image of any prime ideal is prime!

So we define: \mathbb{A}_k^n affine n -space over a field k ,
as a set to be $\{\mathfrak{p} \mid \mathfrak{p} \subset k[y_1, \dots, y_n] \text{ prime}\}$.

Def: R ring (comm. with 1). The spectrum of R as a set is
$$\text{Spec } R := \{ \mathfrak{p} \mid \mathfrak{p} \subset R \text{ prime ideal} \}.$$

Def: The affine space A_R^n as a set is

$$A_R^n := \text{Spec } R[x_1, \dots, x_n]$$

Next, we define a topology on $\text{Spec } R$.

For varieties, the closed sets were the sets of zeros of collections of polynomials (or ideals).

$$\begin{aligned} \text{For } P \in k[y_1, \dots, y_n], \quad P(t_1, \dots, t_n) = 0 \\ \Leftrightarrow P \in (y_1 - t_1, \dots, y_n - t_n) \end{aligned}$$

So, a polynomial vanishes at a maximal or prime ideal \Leftrightarrow it belongs to that ideal

For a general ring, $f \in R$ vanishes at $\mathfrak{p} \in \text{Spec} R$ iff $f \in \mathfrak{p}$.

For an ideal $I \subset R$, to say that the elements of I vanish at \mathfrak{p} means $I \subset \mathfrak{p}$.

Def: R any ring. The Zariski topology on $\text{Spec} R$ is the topology whose closed sets are

$$V(I) := \{ \mathfrak{p} \mid I \subset \mathfrak{p} \} \subset \text{Spec} R$$

for all ideals $I \subset R$.

The functions on $\text{Spec} R$ are given by a sheaf of rings.

Recall that for A_k^n , the ring of functions is

$$k[y_1, \dots, y_n].$$

So, for $\text{Spec} R$, the ring of functions is R .

Given $f \in k[y_1, \dots, y_n]$, $V(f) = \text{zeros of } f$

$U_f := A_k^n \setminus V(f)$ is a basic open set.

On U_f , we can divide by f , so the ring of U_f

is $k[y_1, \dots, y_n] \left[\frac{1}{f} \right] := k[y_1, \dots, y_n, X] / (fX - 1)$

On $\text{Spec} R$, we also have "basic" open sets:

$$U_f := \text{Spec} R \setminus V(f) \quad \text{for } f \in R$$

$$U_f := \{ \mathfrak{p} \mid \mathfrak{p} \subset R, f \notin \mathfrak{p} \}, \quad V(\mathfrak{p}) = \{ \mathfrak{p} \mid \mathfrak{p} \in \mathfrak{p} \}$$

The ring of U_f is, by definition:

$$\mathcal{O}(U_f) := R\left[\frac{1}{f}\right] := R[X] / (fX - 1)$$

Lemma: Any open set in $\text{Spec} R$ is a union of basic open sets. In other words, the basic open sets form a basis of the topology of $\text{Spec} R$.

Proof: $U \subset \text{Spec} R$ open. Then $U = \text{Spec} R \setminus V(\mathfrak{I})$ for some $\mathfrak{I} \subset R$. Choose a set of generators $\{f_j\}_{j \in J}$ for \mathfrak{I} , then $V(\mathfrak{I}) = \bigcap_j V(f_j)$ and $U = \bigcup_j U_{f_j}$. \square

Lemma: $\text{Spec } R$ is quasi-compact.

Proof: Let $\text{Spec } R = \bigcup_{i \in I} V_i$ be an open covering.

By the previous lemma, $\forall i$, we have

$V_i = \bigcup_{j \in J_i} V_{f_{ij}}$ is a union of basic open sets.

$$\text{So } \text{Spec } R = \bigcup_{i,j} V_{f_{ij}}$$

$$\Leftrightarrow \bigcap_{i,j} V(f_{ij}) = \emptyset$$

$$\bigcap_{i,j} V(f_{ij}) = V(\{f_{ij} \mid i,j\}) = V(\langle f_{ij} \mid i,j \rangle)$$

$\bigvee (\langle f_{ij} \mid i, j \rangle) = \phi \Leftrightarrow \nexists \mathfrak{p}$ prime
containing $\langle f_{ij} \mid i, j \rangle$

$$\Leftrightarrow \langle f_{ij} \mid i, j \rangle = R$$

$$\Leftrightarrow 1 \in \langle f_{ij} \mid i, j \rangle$$

$$\Leftrightarrow \exists i_1, \dots, i_n, j_1, \dots, j_n, a_1, \dots, a_n \in R$$

s.t. $1 = \sum_{l=1}^n a_l f_{i_l j_l}$

$$\Rightarrow 1 \in \langle f_{i_1 j_1}, \dots, f_{i_n j_n} \rangle$$

$$\Rightarrow \bigcap_{l=1}^n V(f_{i_l j_l}) = \phi \Rightarrow \text{Spec } R = \bigcup_{l=1}^n V(f_{i_l j_l})$$

$$\Rightarrow \text{Spec } R = \bigcup_{l=1}^n V_{i_l j_l} \quad \square$$

For more details, Eisenbud - Harris: The geometry of schemes.

For an arbitrary open set $U \subset \text{Spec } R$, we write

$$U = \bigcup_{f \in S} U_f \quad \text{for a suitable } S \subset R.$$

and $\mathcal{O}(U) = \varprojlim_{f \in S} \mathcal{O}(U_f)$ inverse limit

We need to define inverse limits and direct limits.

Def: A directed set is a partially ordered set (I, \leq) s.t. $\forall i, j \in I, \exists k \in I$ s.t. $k \geq i, j$

Given a directed set I and a collection of sets $\{S_i \mid i \in I\}$, we define:

Def. Given a collection of maps $\varphi_{ij}: S_i \rightarrow S_j$
 $\forall i \leq j$ s.t. $\forall i \leq j \leq k$ $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$,

the direct limit $\varinjlim_{i \in I} S_i$ is

$$\varinjlim_{i \in I} S_i := \coprod_{i \in I} S_i / \sim$$

where $\alpha_i \sim \beta_j$ if $\exists k \geq i, j$
 $\bigwedge_{i \in I} S_i \quad \bigwedge_{j \in I} S_j$ s.t. $\varphi_{ik}(\alpha_i) = \varphi_{jk}(\beta_j)$.

Def: Given a collection of maps $\varphi_{ij}: S_j \rightarrow S_i$
 $\forall i \leq j$ s.t. $\forall i \leq j \leq k$ $\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$
 we define the inverse limit $\varprojlim_{i \in I} S_i$ to be

$$\varprojlim_{i \in I} S_i := \left\{ (\alpha_i)_{i \in I} \mid \alpha_i \in S_i \quad \forall i \leq j \quad \varphi_{ij}(\alpha_j) = \alpha_i \right\}$$

$$\subset \prod_{i \in I} S_i$$