<u>Def</u>: Given a collection of maps $\varphi_{ij} : S_j \to S_i$ $\forall\, i \le j$ s.t. $\forall\, i \le j \le k$ $\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$ we define the inverse limit $\varprojlim_{i \in I} S_i$ to be

$$\varprojlim_{i \in I} S_i := \left\{ (\alpha_i)_{i \in I} \;\middle|\; \alpha_i \in S_i \;\; \forall\, i \le j \;\; \varphi_{ij}(\alpha_j) = \alpha_i \right\}$$

$$\subset \prod_{i \in I} S_i$$

(strictly speaking, we do not need a directed set here, only a partially ordered set)

As we saw, for $U \subset \operatorname{Spec} R$, we define:

<u>Def</u>: $$\mathcal{O}(U) := \varprojlim_{U_f \subset U} R[f^{-1}]$$

the partial order on the set $\{U_f \mid U_f \subset U\}$ is inclusion.

<u>Note</u>: $f, g \in R$ what does it mean for U_f to be contained in U_g?

$$U_f \subset U_g \iff V(f) \supset V(g)$$

$$\iff \{p \mid p \ni f\} \supset \{p \mid p \ni g\}$$

$$\iff \forall\ p \in \operatorname{Spec} R \quad p \ni g \Rightarrow p \ni f$$

$$\iff \bigcap_{p \ni f} p = \sqrt{\langle f \rangle} \subset \bigcap_{p \ni g} p = \sqrt{\langle g \rangle}$$

$\iff \quad f \in \sqrt{\langle g \rangle}$

$\iff \quad \exists\, n \;\text{ s.t. }\; g \mid f^n$

<u>Exercise</u>: Verify that the definition above for $\mathcal{O}(U)$ indeed defines a sheaf on $\operatorname{Spec} R$. (also see "Geometry of schemes")

We are going to see this is a sheaf in a different way, using the "espace étalé".

<u>Def</u>: The stalk of a presheaf $\mathcal{F}$ on X at a point $x \in X$ (for any topological space X) is

$$\mathcal{F}_x := \varinjlim_{x \in U \subset X} \mathcal{F}(U)$$

More concretely: $\mathcal{F}_x := \coprod_{x \in U \subset X} \mathcal{F}(U) \Big/ \sim$

$$= \Big\{ (U, s) \mid x \in U \subset X,\ s \in \mathcal{F}(U) \Big\} \Big/ \sim$$

where $(U, s) \sim (V, t) \iff \exists_{x \in} W \subset U \cap V$ s.t. $s|_W = t|_W$

Terminology: For $U \subset X$ and $s \in \mathcal{F}(U)$ and $x \in U$, we call the image of (U, s) in $\mathcal{F}_x$ the germ of s at x and we denote it $s(x)$.

Def: The sheaf $\tilde{\mathcal{F}}$ associated to a presheaf $\mathcal{F}$ is the unique (up to isomorphism) sheaf with a morphism

$f : \mathcal{F} \longrightarrow \tilde{\mathcal{F}}$ s.t., for any sheaf $\mathcal{G}$, any morphism of presheaves $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ factors uniquely through $\tilde{\mathcal{F}}$:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\varphi"] \arrow[d, "f"'] & \mathcal{G} \\
\tilde{\mathcal{F}} \arrow[ur, dashed, "\tilde{\varphi}"'] &
\end{tikzcd}$$

$\exists ! \ \tilde{\varphi}$ which makes the diagram commute.

<u>Def</u>: A morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is the data of maps $\varphi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ for all open sets U s.t. $\forall \ V \subset U$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\text{restriction}"'] & \mathcal{G}(U) \arrow[d, "\text{restriction}"] \\
\mathcal{F}(V) \arrow[r, "\varphi(V)"] & \mathcal{G}(V)
\end{tikzcd}$$

A morphism of sheaves is a morphism of the underlying presheaves.

We construct the sheaf associated to a presheaf using the "espace étalé" of a presheaf.

__Def__: The espace étalé of a presheaf $\mathcal{F}$ on X is the set $\overline{\mathcal{F}} := \coprod_{x \in X} \mathcal{F}_x$ with a topology defined below.

$$\begin{array}{ccccc} \overline{\mathcal{F}} & := & \coprod_{x\in X} & \mathcal{F}_x \\ \text{projection } \pi \downarrow & & & \downarrow \\ X & & \ni & x \end{array}$$

Given $U \subseteq X$ any $s \in \mathcal{F}(U)$ defines a section of π:

$$s : U \longrightarrow \coprod_{x \in U} \mathcal{F}_x \subset \overline{\mathcal{F}} \qquad x \longmapsto s(x) \in \mathcal{F}_x$$

We endow $\overline{\mathcal{F}}$ with the topology whose open sets are unions of sets of the form $s(U) \subset \overline{\mathcal{F}}$

<u>Def</u>: The sheaf $\tilde{\mathcal{F}}$ can be defined as the sheaf of continuous sections of π for the above topology.

<u>More concretely</u>: $U \subset X$ open, we describe $\tilde{\mathcal{F}}(U)$.

$$\tilde{\mathcal{F}}(U) := \left\{ f : U \longrightarrow \overline{\mathcal{F}} \mid f \text{ continuous and } \pi \circ f = \mathrm{Id}_U \right\}$$

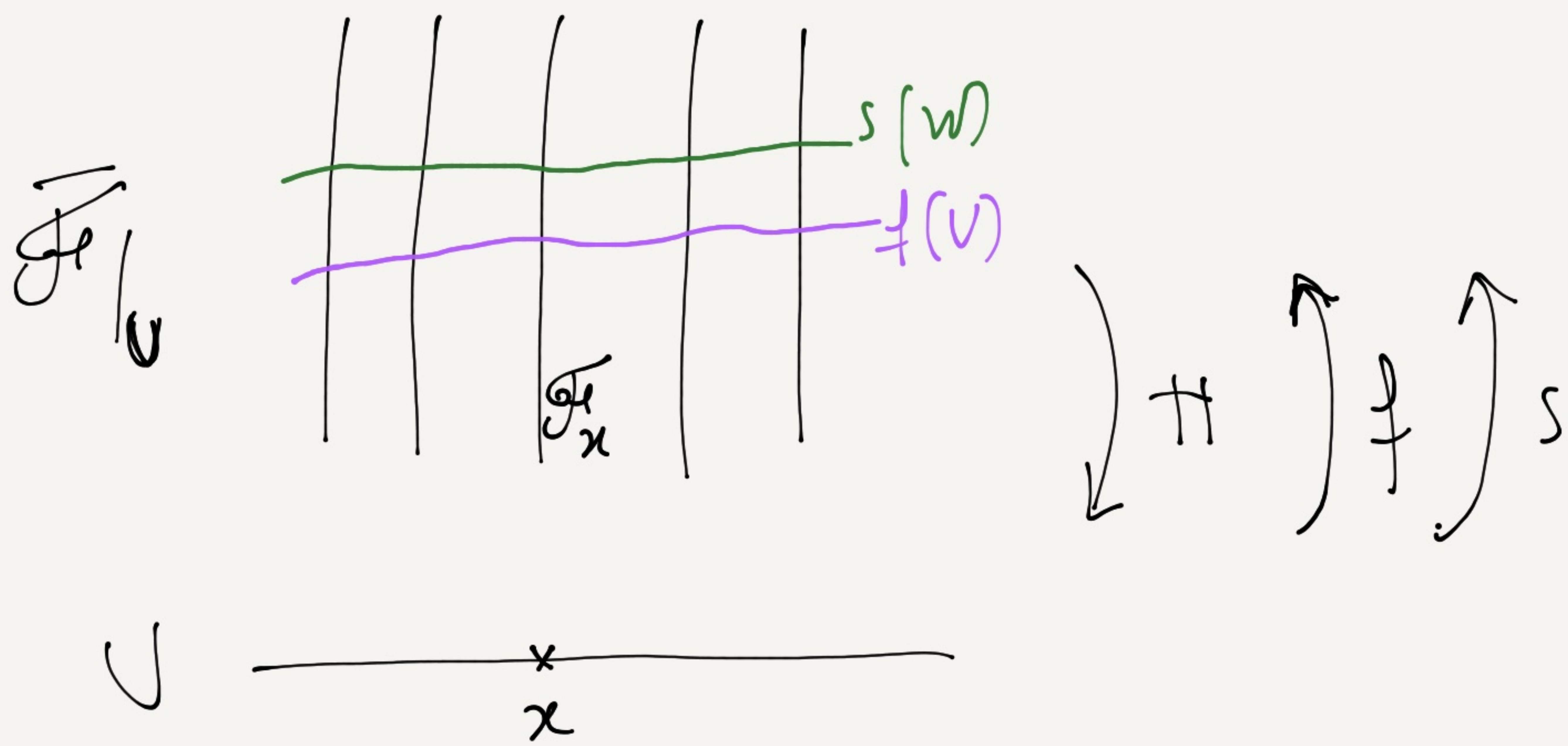
$$= \left\{ f : U \longrightarrow \overline{\mathcal{F}} \mid f \text{ continuous and } \forall x \in U, f(x) \in \mathcal{F}_x \right\}$$

Understanding the continuity: fix $U \subset X$

$f : U \longrightarrow \overline{\mathcal{F}}$ continuous if $\forall\, V \subset \overline{\mathcal{F}}$ open, $f^{-1}(V) \subset U$ is open.

WLOG we can assume $V = s(W)$ for some $W \subset U$ open and $s \in \mathcal{F}(W)$.

What does it mean to say $f^{-1}(s(W))$ is open in U?



$$f^{-1}(s(W)) = \{x \in U \mid f(x) \in s(W)\}$$
$$= \{x \in U \mid f(x) = s(x)\}$$
$$= s^{-1}(f(W))$$