f is continuous $\iff \forall \ x \in W \subset U, \ \forall s \in \mathcal{F}(W)$

$$f^{-1}(s(W)) = \{x \in W \mid f(x) = s(x)\} \text{ is open.}$$

$\iff \forall \ (W, s)$ as above $f^{-1}(s(W))$ is an open neighborhood of all its points.

$$\Updownarrow$$

$\forall \ y \in f^{-1}(s(W)), \ \exists$ open $y \in U_y \subset f^{-1}(s(W))$

$$\Updownarrow$$

$\forall z \in U_y \ f(z) = s(z)$

$$\Updownarrow$$

$\forall \ y \in f^{-1}(s(W)), \ \exists$ open U_y

s.t. $f|_{U_y} = s|_{U_y}$

So f is continuous iff U can be covered with open sets U' s.t. $\forall \ U' \ \exists \ s \in \mathcal{F}(U')$ with $f|_{U'} = s|_{U'}$.

Definition: The morphism of presheaves

$$\mathcal{F} \longrightarrow \tilde{\mathcal{F}}$$

is defined by sending $s \in \mathcal{F}(U)$ to the section of $+1$ defined by s.

Verification of the universal property:

Let $\mathcal{G}$ be a sheaf with a morphism of presheaves

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \\
\tilde{\mathcal{F}} \arrow[ur, dashed, "\tilde{\varphi}"'] &
\end{tikzcd}$$

find $\tilde{\varphi}$ and show it is unique

Uniqueness of $\tilde{\varphi}$:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \\ \tilde{\mathcal{F}} \arrow[ur, "\tilde{\varphi}"'] \arrow[u, phantom, shift left=6, "\circlearrowleft"] & \end{tikzcd}$$

Given $U \subset X$ and $f \in \tilde{\mathcal{F}}(U)$

$\exists$ open covering $U = \bigcup_{i \in I} W_i$ and sections $s_i \in \mathcal{F}(W_i)$

s.t. $f|_{W_i} = s_i \quad \forall i$, i.e., $\forall x \in W_i$, $f(x) = s_i(x)$.

Then $\forall i$ we have $\varphi(s_i) = \tilde{\varphi}(f|_{W_i}) = \tilde{\varphi}(f)|_{W_i}$

So $\tilde{\varphi}(f)|_{W_i}$ is determined by $\varphi(s_i)$

Since $\mathcal{G}$ is a sheaf $\tilde{\varphi}(f)$ is determined

by $\{\tilde{\varphi}(f)|_{W_i} \mid i \in I\} = \{\varphi(s_i) \mid i \in I\}$

$\Rightarrow$ uniqueness.

Existence of $\tilde{\varphi}$: $\quad \tilde{\varphi} : \tilde{\mathcal{F}} \longrightarrow \mathcal{G}$

open $U \subset X$ $\quad f \in \tilde{\mathcal{F}}(U)$, we define $\tilde{\varphi}(f) \in \mathcal{G}(U)$.

$\exists$ open covering $U = \bigcup_{i \in I} W_i$ and $s_i \in \mathcal{F}(W_i)$ s.t.

$\forall i \quad f|_{W_i} = s_i$

So $\tilde{\varphi}(f|_{W_i}) = \varphi(s_i) \Rightarrow \tilde{\varphi}(f)|_{W_i} = \varphi(s_i) \in \mathcal{G}(W_i)$

If we know $\boxed{\varphi(s_i)\big|_{W_i \cap W_j} = \varphi(s_j)\big|_{W_i \cap W_j}}$

$\forall i, j$, then because $\mathcal{G}$ is a sheaf, $\exists!$

section $\tilde{\varphi}(f) \in \mathcal{G}(U)$ s.t. $\tilde{\varphi}(f)\big|_{W_i} = \varphi(s_i)$

$\forall \; x \in W_i \qquad f(x) = s_i(x)$

$\Rightarrow \; \forall x \in W_i \cap W_j \qquad f(x) = s_i(x)$ and $f(x) = s_j(x)$.

$\Rightarrow \; \forall \; x \in W_i \cap W_j \qquad s_i(x) = s_j(x) \in \mathcal{F}_x$.

by the definition of $\mathcal{F}_x$, this means $\exists \; W_x \subset W_i \cap W_j$

s.t. $\qquad s_i|_{W_x} = s_j|_{W_x}$

$\Rightarrow \qquad \forall \; x \in W_i \cap W_j \;, \; \exists \; \underset{\substack{\cup \\ x}}{W_x} \subset W_i \cap W_j$

s.t. $\qquad \varphi(s_i|_{W_x}) = \varphi(s_j|_{W_x})$

$$\varphi(s_i)|_{W_x} = \varphi(s_j)|_{W_x}$$

These W_x form an open covering of $W_i \cap W_j$ and $\mathcal{G}$ is a sheaf, so we can conclude $\quad \varphi(s_i)|_{W_i \cap W_j} = \varphi(s_j)|_{W_i \cap W_j}$. $\square$

Definition (1) **Sheaf of rings:**

A sheaf $\mathcal{O}_X$ on a topological space X is a sheaf of rings if it is a sheaf s.t. $\forall\ U \subset X$, $\mathcal{O}_X(U)$ is a commutative ring with 1 and $\forall\ V \subset U$, the restriction map $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(V)$ is a homomorphism of rings.

(2) **Sheaf of modules over a sheaf of rings:**

Given a sheaf of rings $\mathcal{O}_X$ on X, a sheaf $\mathcal{M}$ of $\mathcal{O}_X$-modules is a sheaf of abelian groups s.t. $\forall$ open $U \subset X$, $\mathcal{M}(U)$ is an $\mathcal{O}_X(U)$-module and $\forall\ V \subset U$, $\forall\ a \in \mathcal{O}_X(U)$, $m \in \mathcal{M}(U)$:

$$(a \cdot m)\big|_V = a\big|_V \cdot m\big|_V .$$

(3) A <u>ringed space</u> is a topological space with a sheaf of rings: $(X, \mathcal{O}_X)$.

(4) A <u>locally ringed space</u> is a ringed space $(X, \mathcal{O}_X)$ s.t. $\forall\, x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.

(5) <u>Push-forward of a sheaf or presheaf:</u>

Given a continuous map $\varphi: Y \to X$ and a presheaf $\mathcal{G}$ on Y, the push-forward $\varphi_*\mathcal{G}$ is the presheaf on X defined by: $\forall$ open $U \subset X$, $(\varphi_*\mathcal{G})(U) := \mathcal{G}(\varphi^{-1}(U))$

<u>exercise:</u> If $\mathcal{G}$ is a sheaf, then so is $\varphi_*\mathcal{G}$.

(6) <u>Morphism of ringed spaces:</u> Given $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ ringed spaces, a morphism $\varphi: (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is

the data of a continuous map $\varphi: Y \to X$

and a morphism of sheaves of rings

$$\varphi^{\#}: \mathcal{O}_X \to \varphi_* \mathcal{O}_Y \quad \text{on } X.$$

Main example: Given a ring R

$(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ is a ringed space (in fact locally ringed space)

Recall, for $f \in R$ $\quad U_f := \{ \mathfrak{p} \mid f \notin \mathfrak{p} \}$
basic open set

$$\mathcal{O}(U_f) := R[f^{-1}] := \cancel{R[x]} \diagup (fx - 1)$$

the U_f form a basis of the topology of $\operatorname{Spec} R$ and

$\forall\, U \subset \operatorname{Spec} R \quad \mathcal{O}(U) = \varprojlim_{U_f \subset U} \mathcal{O}(U_f)$.

We need to define fractions more generally.

<u>Inspiration</u>: $\mathbb{Q} := \mathbb{Z} \times \mathbb{Z} \setminus \{0\} \big/ \sim$

$$(a,b) \sim (c,d) \iff ad - bc = 0$$
$$\left(\frac{a}{b} = \frac{c}{d} \right)$$

<u>Definition</u>: R a ring, a subset $S \subset R$ is called multiplicative if (1) $1 \in S$

(2) $\forall\, s,t \in S$, $st \in S$.

<u>Definition</u>: Given $S \subset R$ multiplicative, define an equivalence relation $\sim$ on $R \times S$ as

$$(a,s) \sim (b,t) \iff \exists\, s' \in S \text{ s.t. } (at - bs)s' = 0$$

The localization $S^{-1}R$ of R at S is

$$S^{-1}R := R \times S \big/ \sim$$

We denote the equivalence class of (a,s) by $\frac{a}{s}$.

There is a natural map
$$R \longrightarrow S^{-1}R$$
$$a \longmapsto \frac{a}{1}$$

<u>Note:</u> If $\exists$ $t \in R$, $s \in S$, s.t. $t \neq 0$ and $ts = 0$, then $\frac{t}{1} = \frac{0}{1}$ in $S^{-1}R$ because $(t\cdot 1 - 0\cdot 1)s = 0$.

<u>Localizations have a universal property:</u>

<u>Prop.:</u> Suppose $S \subset R$ is a multiplicative set and $\varphi: R \to B$ is a morphism of rings s.t. $\forall$ $s \in S$, $\varphi(s)$ is invertible in B. Then $\exists$ a unique homomorphism

$$S^{-1}\varphi : S^{-1}R \longrightarrow B \quad \text{s.t.} \quad S^{-1}\varphi\left(\frac{a}{s}\right) = \varphi(a)\varphi(s)^{-1}$$
$$\forall\ a \in R,\ s \in S$$

i.e., the diagram

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d] & B \\ S^{-1}R \arrow[ur, "S^{-1}\varphi"'] & \end{tikzcd}$$

commutes.

<u>Proof</u>: Atiyah–McDonald, Chapter 3.