In particular, the closed points of $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \dots, x_n]$ are the maximal ideals of $k[x_1, \dots, x_n]$. If k is algebraically closed, we saw that the maximal ideals were all of the form $(x_1 - a_1, \dots, x_n - a_n)$ for some $(a_1, \dots, a_n) \in k^n$.

So, if k is algebraically closed, then we can think of k^n as the set of closed points of $\mathbb{A}^n_k$.

Back to general sheaves and schemes:

We saw that morphisms of (pre)sheaves induce morphisms on the stalks.

Lemma: (see text) A morphism of sheaves is an isomorphism if and only if it induces isomorphisms on all the stalks.

In homework you will see that a morphism of sheaves

is $\begin{cases} \text{injective} \\ \text{surjective} \end{cases}$ $\iff$ it is $\begin{cases} \text{injective on all the stalks} \\ \text{surjective} \end{cases}$

In the text: Lemma: Given two rings A, B, the datum of a morphism $\operatorname{Spec} A \longrightarrow \operatorname{Spec} B$ is equivalent to the datum of a homomorphism of rings $B \longrightarrow A$.

We will prove the stronger:

Lemma: Suppose given a locally ringed space X and a ring A. The natural map

$$\operatorname{Hom}(X, \operatorname{Spec} A) \longrightarrow \operatorname{Hom}(A, \mathcal{O}_X(X))$$

obtained by sending $\varphi : X \longrightarrow \operatorname{Spec} A$ to its "global sections"

$$\varphi^\#(\operatorname{Spec} A) : \mathcal{O}_A(\operatorname{Spec} A) \longrightarrow (\varphi_* \mathcal{O}_X)(\operatorname{Spec} A)$$

$$\begin{array}{ccc} \mathcal{O}_A(\operatorname{Spec} A) & \longrightarrow & (\varphi_* \mathcal{O}_X)(\operatorname{Spec} A) \\ \| & & \| \\ & & \mathcal{O}_X(\varphi^{-1}(\operatorname{Spec} A)) \\ & & \| \\ \varphi^\#(\operatorname{Spec} A) : A & \longrightarrow & \mathcal{O}_X(X) \end{array}$$

is a bijection.

<u>Proof</u>: We prove surjectivity first: given a hom. of rings $\alpha : A \longrightarrow \mathcal{O}_X(X)$, we construct a morphism $\varphi : X \longrightarrow \operatorname{Spec} A$ s.t.

$$\alpha = \varphi^\#(\operatorname{Spec} A).$$

We first define the map of sets: $\varphi : X \to \operatorname{Spec} A$.

Let $x \in X$, $\varphi(x)$?

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[dr, "\gamma_x \circ \alpha"'] & \mathcal{O}_X(X) \arrow[d, "\gamma_x"] & \ni s \arrow[d, mapsto] \\ & \mathcal{O}_{X,x} & \ni s(x) \end{tikzcd}$$

$\mathcal{O}_{X,x} \supset \mathfrak{m}_x$ unique maximal ideal.

<u>Terminology</u>: γ_x is called the evaluation map at x.

Define: $\varphi(x) := (\gamma_x \circ \alpha)^{-1}(\mathfrak{m}_x) \subset A$

prime ideal of A.

We show φ is continuous: $\forall f \in A$, $\varphi^{-1}(U_f) \subset X$ is open.

$$U_f = \{ \mathfrak{p} \subset A \mid f \notin \mathfrak{p} \}$$

$$\varphi^{-1}(U_f) = \{ x \in X \mid \varphi(x) \in U_f \}$$

$$= \{ x \in X \mid (\gamma_x \circ \alpha)^{-1}(m_x) \in U_f \}$$

$$= \{ x \in X \mid f \notin (\gamma_x \circ \alpha)^{-1}(m_x) \}$$

$$= \{ x \in X \mid (\gamma_x \circ \alpha)(f) \notin m_x \}$$

$$= \{ x \in X \mid (\gamma_x \circ \alpha)(f) \in \mathcal{O}_{X,x} \text{ is invertible} \}$$

$$= \{ x \in X \mid \exists\, h_x \in \mathcal{O}_{X,x} \text{ s.t. } h_x \cdot (\gamma_x \circ \alpha)(f) = 1 \}$$

$$= \{ x \in X \mid \exists\, U \subset X \text{ open neighborhood of } x,\ \exists\, h \in \mathcal{O}_X(U) \text{ s.t. } h(x) \cdot (\gamma_x \circ \alpha)(f) = 1 \}$$

__Note:__ $(\gamma_x \circ \alpha)(f) = \gamma_x(\alpha(f)) = (\alpha(f))(x)$ the germ of $\alpha(f)$ at x

$$\Rightarrow \quad \varphi^{-1}(U_f) = \{ x \in X \mid \exists\ V \subset X \text{ open neighborhood of } x,\ \exists\ h \in \mathcal{O}_X(V) \text{ s.t. } h(x) \cdot (\alpha(f))(x) = 1 \}$$

<u>Note:</u> $h(x) \cdot (\alpha(f))(x) = h(x) \left(\alpha(f)|_V\right)(x)$

$$\begin{tikzcd} f \in A \arrow[r] & \mathcal{O}_X(X) \arrow[r] \arrow[rr, bend right, "\gamma_x"'] & \mathcal{O}_X(V) \arrow[r] & \mathcal{O}_{X,x} \end{tikzcd}$$

$$\Rightarrow \quad h(x) \cdot (\alpha(f))(x) = \left(h \cdot \alpha(f)|_V\right)(x) = 1$$

$\Rightarrow \quad \exists \quad W \subset V$ open neighborhood of x

s.t. $\quad \left(h \cdot \alpha(f)|_V\right)|_W = 1$

$$\parallel$$

$$h|_W \cdot \alpha(f)|_W = 1$$

$$\varphi^{-1}(U_f) = \left\{ x \in X \;\middle|\; \begin{array}{l} \exists\ V_x \subset X \text{ open neighborhood of } x \\ \exists\ h \in \mathcal{O}_X(V_x) \text{ s.t. } h|_{V_x} \cdot \alpha(f)|_{V_x} = 1 \end{array} \right\}$$

$\forall\ x \in \varphi^{-1}(U_f)$, $\varphi^{-1}(U_f) \supset V_x$ because

$$\forall y \in V_x \qquad h(y)\cdot(\alpha(f))(y) = 1$$

$\Rightarrow\ \varphi^{-1}(U_f)$ is open.

Next we define the morphism of sheaves

$$\varphi^{\#} : \mathcal{O}_A \longrightarrow \varphi_* \mathcal{O}_X \quad \text{on } \operatorname{Spec} A$$

we already have $\varphi^{\#}(\operatorname{Spec} A) : A \longrightarrow (\varphi_* \mathcal{O}_X)(\operatorname{Spec} A)$

$$A \xrightarrow{\alpha} \mathcal{O}_X(X) = (\varphi_* \mathcal{O}_X)(\operatorname{Spec} A)$$

We first define $\varphi^{\#}$ on basic open sets:

choose $f \in A$ $\qquad \mathcal{O}_A(U_f) = A[f^{-1}]$

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d] \arrow[dr, "\alpha|_{U_f}"] & \mathcal{O}_X(X) = (\varphi_*\mathcal{O}_X)(\operatorname{Spec} A) \arrow[d] \\
A[f^{-1}] \arrow[r, dotted, "?"] & \mathcal{O}_X(\varphi^{-1}(U_f)) = (\varphi_*\mathcal{O}_X)(U_f)
\end{tikzcd}$$

<u>recall</u>: $\varphi^{-1}(U_f) = \{x \in X \mid \alpha(f)(x) \in \mathcal{O}_{X,x} \text{ is invertible}\}$

$$\begin{array}{c} \| \\ X_{\alpha(f)} \end{array}$$

We need to show that $\alpha|_{U_f}$ factors through $A[f^{-1}]$.

By the universal property of localizations, we need to show that $\alpha(f)|_{\varphi^{-1}(U_f) = X_{\alpha(f)}}$ is invertible in $\mathcal{O}_X(\varphi^{-1}(U_f))$.

$$\mathcal{O}_X(\varphi^{-1}(U_f)) = \mathcal{O}_X(X_{\alpha(f)})$$

We saw that $\forall\ x \in X_{\alpha(f)} = \varphi^{-1}(U_f)$

$\exists\ V_x \ni x$ s.t. $\alpha(f)|_{V_x}$ is invertible.

$\Rightarrow$ $\exists$ covering $X_{\alpha(f)} = \bigcup_{i\in I} V_i$ $\forall i\ \exists\ h_i \in \mathcal{O}_X(V_i)$

s.t. $\alpha(f)|_{V_i} \cdot h_i = 1$

<u>Claim:</u> the h_i glue together to an inverse for $\alpha(f)|_{X_{\alpha(f)}}$.

$$h_i|_{V_i\cap V_j} \cdot \alpha(f)|_{V_i\cap V_j} = 1$$

and $$h_j|_{V_i\cap V_j}\ \alpha(f)|_{V_i\cap V_j} = 1$$

$\Longrightarrow$ uniqueness of inverses $$h_i|_{V_i\cap V_j} = h_j|_{V_i\cap V_j}$$

$\Rightarrow \quad \exists!\ h \in \mathcal{O}_X(X_{\alpha(f)})$ s.t. $h\, \alpha(f)|_{X_{\alpha(f)}} = 1$.

$\Rightarrow \quad \alpha(f)|_{X_{\alpha(f)} = \varphi^{-1}(U_f)}$ is invertible

$\Rightarrow$ we have a factorization (unique)

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "?"] & \mathcal{O}_X(X) \arrow[d] \\ A[f^{-1}] \arrow[r, "\alpha_f"'] & \mathcal{O}_X(X_{\alpha(f)}) \end{tikzcd}$$

To define $\varphi^{\#}$ for arbitrary open sets, we pass to the inverse limit: $\mathcal{O}_A(U) = \varprojlim_{U_f \subset U} \mathcal{O}_A(U_f)$

In order for this to work, we need compatibility with the restriction morphisms, i.e., we need to show that whenever $U_g \subset U_f$, we have a commutative diagram:

$$
\begin{tikzcd}
f, g \in A_h \arrow[r, "\alpha"] \arrow[d] & \mathcal{O}_X(X_{\alpha(h)}) \arrow[d] \\
\frac{f}{1}, \frac{g}{1} \in A[f^{-1}] \arrow[r, "\alpha_f"] \arrow[d] & \mathcal{O}_X(X_{\alpha(f)}) \arrow[d] \\
A[f^{-1}]\left[\left(\tfrac{g}{1}\right)^{-1}\right] \overset{\text{universal property}}{=} A[g^{-1}] \arrow[r, "\alpha_g"] & \mathcal{O}_X(X_{\alpha(g)})
\end{tikzcd}
$$

$$\frac{f}{h} \longmapsto \alpha(f)\alpha(h)^{-1}$$

follows from the universal property of localizations. or the explicit description of α_f, α_g.

<u>Recall</u>: what it means for U_f to contain U_g:

$$U_g \subset U_f$$

$$\{p \not\ni g\} \subset \{p \not\ni f\}$$

$$\iff \{p \ni g\} \supset \{p \ni f\} \iff \sqrt{(f)} \ni g$$

$$\iff \exists\, n > 0,\ \exists\, a \in A \text{ s.t. } g^n = af$$

So we have $\varphi^\# : \mathcal{O}_{\operatorname{Spec} A} \longrightarrow \varphi_* \mathcal{O}_X$

$\forall\ x \in X \qquad \varphi_x : \mathcal{O}_{A,x} \longrightarrow \mathcal{O}_{X,x}$

recall $\varphi(x) = \alpha^{-1}(m_x) \subset A$

$\in \operatorname{Spec} A$

$$\begin{tikzcd}
\varphi(x) = \alpha^{-1}(m_x) \\
\mathfrak{p} \subset A \arrow[r] \arrow[d] & \mathcal{O}_X(X) \arrow[d] \\
A_{\mathfrak{p}} = \mathcal{O}_{A,x} \arrow[r, "\varphi_x"] & \mathcal{O}_{X,x} \\
\mathfrak{p}A_{\mathfrak{p}} \text{ maximal ideal}
\end{tikzcd}$$

$\Rightarrow \boxed{\varphi_x^{-1}(m_x) = \mathfrak{p} A_{\mathfrak{p}}}$

$\Rightarrow \varphi : X \longrightarrow \operatorname{Spec} A$ is a morphism of locally ringed space.