Injectivity: If two morphisms $\varphi, \psi : X \longrightarrow \operatorname{Spec} A$ induce the same map $\alpha : A \longrightarrow \mathcal{O}_X(X)$, then $\varphi = \psi$.

$$\varphi^{\#}(\operatorname{Spec} A) = \psi^{\#}(\operatorname{Spec} A) = \alpha : A \longrightarrow \mathcal{O}_X(X)$$

For $x \in X$, put $\mathfrak{p} := \varphi(x)$, $\mathfrak{q} := \psi(x)$

$$\begin{tikzcd}
 & A \arrow[rr] \arrow[dl, "\gamma_{\mathfrak{p}}"'] \arrow[dd, "\gamma_{\mathfrak{q}}"] \arrow[drr, "\gamma_x \circ \alpha"] & & \mathcal{O}_X(X) \arrow[d, "\gamma_x"] \\
A_{\mathfrak{p}} \arrow[rrr, "\varphi_x"] & & & \mathcal{O}_{X,x} \\
 & A_{\mathfrak{q}} \arrow[urr, "\psi_x"'] & &
\end{tikzcd}$$

we know:

$$\varphi_x^{-1}(\mathfrak{m}_x) = \mathfrak{p} A_{\mathfrak{p}}$$

$$\psi_x^{-1}(\mathfrak{m}_x) = \mathfrak{q} A_{\mathfrak{q}}$$

The diagrams commute: $\gamma_x \circ \alpha = \varphi_x \circ \gamma_{\mathfrak{p}} = \psi_x \circ \gamma_{\mathfrak{q}}$

$$\Longrightarrow \quad (\gamma_x \circ \alpha)^{-1}(\mathfrak{m}_x) = (\varphi_x \circ \gamma_{\mathfrak{p}})^{-1}(\mathfrak{m}_x) = (\psi_x \circ \gamma_{\mathfrak{q}})^{-1}(\mathfrak{m}_x)$$

$$(\varphi_x \circ \gamma_{\mathfrak{p}})^{-1}(\mathfrak{m}_x) = \gamma_{\mathfrak{p}}^{-1}(\varphi_x^{-1}(\mathfrak{m}_x)) = \gamma_{\mathfrak{p}}^{-1}(\mathfrak{p} A_{\mathfrak{p}}) = \mathfrak{p}$$

Similarly $(\varphi_x \circ \gamma_{\mathfrak{q}})^{-1}(m_x) = \mathfrak{q}$

$\Rightarrow \quad \mathfrak{p} = \mathfrak{q}$, i.e., $\varphi(x) = \psi(x) \quad \forall x \in X$.

So $\varphi = \psi : X \longrightarrow \operatorname{Spec} A$ as maps of spaces.

The maps $\varphi^{\#}$ and $\psi^{\#}$ are also equal on all basic open sets U_f, as maps $A[f^{-1}] \longrightarrow \mathcal{O}_X(X_{\alpha(f)})$ because they are obtained by localizing α. $\square$

We generalized affine varieties: $Y \subset k^n$, $Y = V(I)$, $I \subset k[y_1, \dots, y_n] = A$

$A(Y) = A/I(Y)$

$Y \longleftrightarrow$ scheme $\operatorname{Spec} A(Y)$

What do we do about projective varieties? We make everything homogeneous!

The Proj of a graded ring:

Recall: A graded ring S (see Atyiah-McDonald) is a commutative ring with 1 and a direct sum decomposition (as abelian groups)

$$S = \bigoplus_{\substack{d\in\mathbb{Z} \\ d\geq 0}} S_d$$

s.t. $\quad S_d \cdot S_e \subset S_{d+e} \quad \forall\ d,e$

Note: $S_0 \subset S$ is a subring.

Recall: $S_+ := \bigoplus_{d>0} S_d$ is an ideal

Definition: As a set $\operatorname{Proj} S := \left\{ \mathfrak{p} \,\middle|\, \begin{array}{l} \mathfrak{p} \subset S \text{ homogeneous prime ideal} \\ \mathfrak{p} \not\supset S_+ \end{array} \right\}$

The closed sets of the topology on Proj S are

$$Z(I) := \{ \mathfrak{p} \in \operatorname{Proj} S \mid \mathfrak{p} \supset I \} \subset \operatorname{Proj} S$$

for all $I \subset S$ homogeneous ideal

We have basic open sets: $\forall f \in S$ homogeneous

$$U_f := \{ \mathfrak{p} \in \operatorname{Proj} S \mid \mathfrak{p} \not\ni f \} \subset \operatorname{Proj} S$$

These form a basis of the topology, similarly to the affine case.

For a basic open set U_f, we define the ring of U_f:

$$\mathcal{O}_{\operatorname{Proj} S}(U_f) := S[f^{-1}]_0$$ the subring of degree 0 elements in $S[f^{-1}]$

(intuitively: dehomogenization: $k\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] \cong \left(k[x_0, \dots, x_n][x_i^{-1}]\right)_0$)

For arbitrary open sets $V \subset \operatorname{Proj} S$

$$\mathcal{O}_{\operatorname{Proj} S}(V) = \varprojlim_{U_f \subset V} \mathcal{O}_{\operatorname{Proj} S}(U_f) = \varprojlim_{U_f \subset V} S[f^{-1}]_0$$

<u>Lemma:</u> $\forall\ p \in \operatorname{Proj} S$, we have a canonical isomorphism $\mathcal{O}_{\operatorname{Proj} S, p} \cong S_{p,0}$ the subring of elements of degree 0 of S_p.

<u>Proof:</u> Taking direct limits commutes with elements of degree 0 (and use the result in the affine case). $\square$

<u>Corollary:</u> $(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S})$ is a locally ringed space.

<u>Recall:</u> Any open subset $U \subset X$ of a (locally) ringed space inherits a structure of (locally) ringed space:

$V \subset U \subset X \qquad \mathcal{O}_U(V) := \mathcal{O}_X(V)$.

<u>Lemma:</u> For all homogeneous elements $f \in S$, we have $U_f \cong \operatorname{Spec} S[f^{-1}]_0$ as locally ringed spaces.

<u>Proof:</u> We have $\mathcal{O}_{\operatorname{Proj} S}(U_f) \xleftarrow{=} S[f^{-1}]_0$.

By the lemma we proved in the las lecture, this defines a morphism $U_f \xrightarrow{\varphi} \operatorname{Spec} S[f^{-1}]_0$.

We need to show this is an isomorphism.

<u>First</u>: show it is a bijection on the sets:

$\mathfrak{q} \in U_f \qquad \mathfrak{q} \subset S$ homogeneous prime, $f \notin \mathfrak{q}$.

$$\varphi(\mathfrak{q}) = (\gamma_{\mathfrak{q}})^{-1}\left((\mathfrak{q} S_{\mathfrak{q}})_0\right) = \mathfrak{q} S[f^{-1}]_0$$

$$\begin{tikzcd}
S[f^{-1}]_0 \arrow[r, "\cong"] \arrow[dr] & \mathcal{O}_{\operatorname{Proj} S}(U_f) \arrow[d, "\gamma_{\mathfrak{q}}"] \\
 & \mathcal{O}_{\operatorname{Proj} S, \mathfrak{q}} = S_{\mathfrak{q}, 0}
\end{tikzcd}$$

injectivity of φ means

$$\forall\ \mathfrak{q}_1, \mathfrak{q}_2, \quad \mathfrak{q}_1 S[f^{-1}]_0 = \mathfrak{q}_2 S[f^{-1}]_0 \Rightarrow \mathfrak{q}_1 = \mathfrak{q}_2$$

surjectivity means $\forall$ prime ideal $\mathfrak{q}_0 \subset S[f^{-1}]_0$

$\exists\ \mathfrak{q} \subset S$ homogeneous prime with $f \notin \mathfrak{q}$

s.t. $\mathfrak{q}_0 = \mathfrak{q}\, S[f^{-1}]_0$ to be discussed later.

<u>Reference</u>: Vakil: "The rising sea" p. 149. $\square$

Definition: Given a ring R, we define projective n-space over R as $\mathbb{P}^n_R := \operatorname{Proj} R[x_0, \dots, x_n]$

We can also generalize projective varieties!

$$Y = Z(I) \qquad I = I(Y) \text{ homogeneous ideal}$$

$$S(Y) = S/I \qquad S = R[x_0, \dots, x_n]$$

$$Y = \operatorname{Proj} S(Y).$$

Next we study general schemes a little more.