First properties of schemes :

<u>Def</u>: (1) A scheme is connected/irreducible/quasi-compact if its underlying topological space is connected/irreducible/quasi-compact.

(2) A scheme is reduced if $\forall$ open sets $U \subset X$, the ring $\mathcal{O}_X(U)$ is reduced, i.e., has no nilpotents ($a \in A$ is nilpotent if $a \neq 0$ and $\exists n$ $a^n = 0$)

Equivalently (homework), for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is reduced.

(3) A scheme is integral if, for all non-empty open sets $U \subset X$ ring $\mathcal{O}_X(U)$ is an integral domain ($\neq 0$).

(4) A scheme is locally noetherian if it can be covered by

open affine sets $U \cong \operatorname{Spec} A$ with A noetherian.

(5) A scheme is noetherian if it is locally noetherian and quasi-compact. Exercise: Equivalently, it has a finite cover by open affine sets with noetherian rings of sections.

(recall that affine schemes are quasi-compact)

<u>Prop.</u>: A scheme is integral if and only if it is irreducible and reduced.

<u>Proof</u>: Suppose X is an integral scheme.
Clearly, X is reduced. We show X is irreducible.

If $X = X_1 \cup X_2$ with X_1, X_2 closed,
put $U_i = X \setminus X_i$: this is open.

$$U_1 \cap U_2 = (X \setminus X_1) \cap (X \setminus X_2) = X \setminus (X_1 \cup X_2) = \emptyset$$

The sheaf property implies that the map

$$\mathcal{O}_X(U_1 \amalg U_2) \longrightarrow \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$$

product of $|_{U_1}$ and $|_{U_2}$

is an isomorphism of rings.

$\mathcal{O}_X(U_1 \amalg U_2)$ is an integral domain, so

$$(1,0)\cdot(0,1) = 0 \implies (1,0) = 0 \text{ or } (0,1) = 0$$

$$\implies \mathcal{O}_X(U_1) = 0 \text{ or } \mathcal{O}_X(U_2) = 0$$

$$\implies U_1 = \emptyset \text{ or } U_2 = \emptyset$$ (because X is integral, the only open set with ring 0 is the empty set)

$$\implies X = X_1 \text{ or } X = X_2$$

Conversely, suppose X is reduced and irreducible.

Let $U \subset X$ be open, $U \neq \phi$.

Suppose $f, g \in \mathcal{O}_X(U)$ with $fg = 0$.

Recall: For a locally ringed space $(X, \mathcal{O}_X)$,
$\forall f \in \mathcal{O}_X(X)$, the set $X_f := \{x \in X \mid f(x) \notin m_x \subset \mathcal{O}_{X,x}\}$
is open.

endow U with the locally ringed space structure induced from X: $V \subset U$ $\mathcal{O}_U(V) := \mathcal{O}_X(V)$.

Then $Z(f) := \{x \in U \mid f(x) \in m_x \subset \mathcal{O}_{U,x}\}$
is closed in U $= U \setminus U_f$

Similarly $Z(g)$ is closed in U.

$fg = 0 \Rightarrow Z(fg) = U \Rightarrow Z(f) \cup Z(g) = U$

Recall from homework that U is irreducible.

$$\Rightarrow \quad Z(f) = U \quad \text{or} \quad Z(g) = U$$

Suppose $Z(f) = U$. Then for any affine open $\operatorname{Spec} A \subset U$,

$$\forall\ \mathfrak{p} \in \operatorname{Spec} A \qquad \mathcal{O}_{U,\mathfrak{p}} = A_{\mathfrak{p}}$$

$$\text{and} \quad m_{\mathfrak{p}} = \mathfrak{p} A_{\mathfrak{p}}$$

$$Z(f) = U \Rightarrow Z(f) \supset \operatorname{Spec} A \Rightarrow f(\mathfrak{p}) \in \mathfrak{p} A_{\mathfrak{p}} \quad \forall\ \mathfrak{p} \in \operatorname{Spec} A$$

$f(\mathfrak{p}) =$ image of $f|_{\operatorname{Spec} A}$ in the localization $A_{\mathfrak{p}}$

$$= \frac{f|_{\operatorname{Spec} A}}{1}$$

$$\Rightarrow \quad \forall\ \mathfrak{p} \in \operatorname{Spec} A \qquad f|_{\operatorname{Spec} A} \in \mathfrak{p} \subset A$$

$$\mathcal{O}_X(U) \to A$$

$\Rightarrow \quad f|_{\operatorname{Spec} A} \in \bigcap_{p \in \operatorname{Spec} A} p$ = nilradical

:= set of all nilpotent elements

$\Rightarrow \quad f|_{\operatorname{Spec} A}$ is nilpotent $\in A$

we assumed X is reduced $\Rightarrow \quad f|_{\operatorname{Spec} A} = 0$

This holds for all open affine in $U \Rightarrow f = 0$

because U is covered by open affine subsets.

$\Rightarrow \quad \mathcal{O}_X(U)$ is an integral domain.

$\square$

<u>Proposition:</u> A scheme is locally noetherian if and only if, for any open affine $\operatorname{Spec} A \cong U \subset X$, A is noetherian. In particular, an affine scheme is locally

noetherian if and only if A is noetherian.

<u>Proof</u>: The "if" part is clear.

Assume X is locally noetherian, let $\operatorname{Spec} A = U \subset X$ be open. We will show that A is a noetherian ring.

X has a cover by open affine sets. $X = \bigcup_{i \in I} \operatorname{Spec} B_i$

Choose $i \in I$

$\forall \; f \in B = B_i \quad B[f^{-1}] = B[x]/(fx-1)$ is noetherian by the Hilbert basis theorem.

$\Rightarrow$ open affine sets with noetherian rings of sections form a basis of the topology of X.

$\Rightarrow$ $U = \operatorname{Spec} A$ has a covering by open sets with noetherian rings of sections.

$U = \bigcup_{j \in J} V_j \qquad V_j = \operatorname{Spec} C_j \qquad C_j$ noetherian

$\exists\, f \in A$ s.t. $\quad U_f = \operatorname{Spec} A[f^{-1}] \subset V_j = \operatorname{Spec} C_j$

write $V_j = V$, $C_j = C$

$$U \supset V \qquad \begin{array}{ccc} A = \mathcal{O}(U) & \longrightarrow & C = \mathcal{O}(V) \\ \rotatebox{90}{$\in$} & & \\ f & \longmapsto & f|_V =: \bar{f} \end{array}$$

Claim : $\quad U_f = V_{\bar{f}} \quad \left(\Longrightarrow \underset{\text{ring of } U_f}{A[f^{-1}]} = \underset{\text{ring of } V_{\bar{f}}}{C[\bar{f}^{-1}]} \right)$

$$U_f = \{ p \in \operatorname{Spec} A \mid f \notin p \} = \{ p \in \operatorname{Spec} A \mid f(p) \notin pA_p \}$$
$$= \{ q \in \operatorname{Spec} C \mid \bar{f}(q) \notin q C_q \}$$
$$= \{ q \in \operatorname{Spec} C \mid \bar{f} \notin q \} = V_{\bar{f}} \subset \operatorname{Spec} C$$

U is affine, hence quasi-compact, so we can cover U with a finite number of open sets $U_{f_1}, \dots, U_{f_n}$ whose rings are $A[f_1^{-1}] = C_{j_1}[f_1^{-1}], \dots, A[f_n^{-1}] = C_{j_n}[f_n^{-1}]$ which are noetherian.

Since $U = U_{f_1} \cup \cdots \cup U_{f_n}$, the elements $f_1, \dots, f_n$ generate the unit ideal in A.

Now we show A is noetherian.

Let $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots \subset \mathfrak{a}_n \subset \cdots$ be an ascending chain of ideals. $\forall i$

$$\mathfrak{a}_1[f_i^{-1}] \subset \mathfrak{a}_2[f_i^{-1}] \subset \cdots \subset A[f_i^{-1}]$$

is stationary because $A[f_i^{-1}]$ is noetherian.

The original chain $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots$ is stationary by the following lemma:

<u>Lemma</u>: Let A be a ring, $f_1, \dots, f_n \in A$ which generate the unit ideal. Then for any ideal $\mathfrak{a} \subset A$,

$$\mathfrak{a} = \bigcap_{i=1}^{n} \varphi_i^{-1}\left(\varphi_i(\mathfrak{a}) A[f_i^{-1}]\right)$$

where $\varphi_i : A \longrightarrow A[f_i^{-1}]$ is the localization morphism.

<u>Proof</u>: Clearly $\mathfrak{a} \subset \bigcap_{i=1}^{n} \varphi_i^{-1}\left(\varphi_i(\mathfrak{a}) A[f_i^{-1}]\right)$

For the reverse inclusion, choose $b \in \bigcap_{i=1}^{n} \varphi_i^{-1}\left(\varphi_i(\mathfrak{a}) A[f_i^{-1}]\right)$

$\forall i \quad \exists \; n_i \in \mathbb{Z}, \exists \; a_i \in \mathfrak{a}$ s.t. $\varphi_i(b) = \frac{a_i}{f_i^{n_i}} = \frac{f_i^n a_i}{f_i^{n_i+n}} \quad \forall n$

$\Rightarrow$ can assume $\exists\ n$, $a_i \in \mathcal{O}_b$ s.t. $\forall i$ $\varphi_i(b) = \dfrac{a_i}{f_i^n}$

$$\frac{b}{1} = \varphi_i(b) = \frac{a_i}{f_i^n}$$

$\exists\ m_i \in \mathbb{Z}_+$ s.t. $\quad f_i^{m_i}(f_i^n b - a_i) = 0$

increase m_i if necessary:

$\exists\ m, n$ s.t. $\quad f_i^m (f_i^n b - a_i) = 0$

$$\Rightarrow \quad \forall i \quad f_i^{m+n} b \ (= f_i^m a_i) \in \mathcal{O}_b$$

$$\langle f_1, \ldots, f_r \rangle = A \quad \Rightarrow \quad \langle f_1^{m+n}, \ldots, f_r^{m+n} \rangle = A$$

$\Rightarrow \quad \exists\ c_1, \ldots, c_r \in A$ s.t. $\quad 1 = c_1 f_1^{m+n} + \cdots + c_r f_r^{m+n}$

$$b = b \cdot 1 = c_1 f_1^{m+1} b_1 + \cdots + c_r f_r^{m+n} b \in \mathcal{O}_b.$$

$\square$