For simplicity, assume k is algebraically closed

In $Z \times_k Z$, we have 4 nonseparated points:

(recall: in this case, the set of closed points of $Z \times_k Z$ the product of the set of closed points of Z with itself.)

$$(O_1, O_1),\ (O_1, O_2),\ (O_2, O_1),\ (O_2, O_2)$$

The image of $\Delta : Z \longrightarrow Z \times_k Z$ contains $(O_1, O_1), (O_2, O_2)$ but not (O_1, O_2) and (O_2, O_1) which are in its closure.

Details: exercise: hint: construct the fiber product:

$$Z \times_k Z = X \times_k X \cup X \times_k Y \cup Y \times_k X \cup Y \times_k Y$$

Lemma: A morphism $f : X \longrightarrow Y$ is separated if and only if the image of Δ is a closed subset of $X \times_Y X$.

Proof: If Δ is a closed embedding, its image is a closed subset of $X \times_Y X$.

Conversely, assume the image of Δ is closed.

$$
\begin{tikzcd}
X \arrow[dr, "\Delta"] \arrow[drr, bend left, "Id_X"] \arrow[ddr, bend right, "Id_X"'] & & \\
& X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\mathrm{I}"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

$\Delta(X) \subset X \times_Y X$ is closed.

Claim: $\Delta : X \longrightarrow \Delta(X)$ is a homeomorphism

because $p_1 \circ \Delta = \mathrm{Id}_X$.

Next we need to see that $\Delta^\# : \mathcal{O}_{X \times_Y X} \longrightarrow \Delta_* \mathcal{O}_X$

is surjective. From past homework, we know that

$\Delta^\#$ is surjective iff it is surjective on the stalks.

Choose $x \in X$ $\quad \Delta^\#(x) : \mathcal{O}_{X \times X, \Delta(x)} \longrightarrow \mathcal{O}_{X,x}$

Choose an open affine neighborhood $U = \operatorname{Spec} A$ of x.

Suppose U small enough so that $f(U) \subset \underset{\text{open}}{\operatorname{Spec} B} \subset Y$

$U \times_V U \subset X \times_Y X$ is an open affine neighborhood of $\Delta(x)$.

We saw that $\Delta|_U : U \longrightarrow U \times_Y U$ is a closed embedding (every morphism of affine schemes is separated)

$$\mathcal{O}_{U\times_Y U,\Delta(x)} = \mathcal{O}_{X\times_Y X,\Delta(x)}$$

$$\mathcal{O}_{U,x} = \mathcal{O}_{X,x}$$

$\Rightarrow$

$$\begin{tikzcd}
\mathcal{O}_{U\times_Y U,\Delta(x)} \arrow[r] \arrow[d, equal] & \mathcal{O}_{U,x} \arrow[d, equal] \\
\mathcal{O}_{X\times_Y X,\Delta(x)} \arrow[r] & \mathcal{O}_{X,x}
\end{tikzcd}$$

$\mathcal{O}_{U\times_Y U,\Delta(x)} \to \mathcal{O}_{U,x}$ is surjective because $\Delta|_U$ is a closed embedding

$\square$

Valuative criterion: Uses valuation rings. (Atiyah-McDonald)

Def: K a field. A subring R of K is called a valuation ring if $\forall\ x \neq 0,\ x \in K$, either $x \in R$ or $x^{-1} \in R$.

Main results about valuation rings:

① Valuation rings are local rings.

② Given two local subrings $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ of K, we say $(A, \mathfrak{m})$ dominates $(B, \mathfrak{n})$ if $B \subset A$ and

$$\mathfrak{m} \cap B = \mathfrak{n}.$$

This defines a partial order among local subrings of K. Valuation rings are exactly the maximal elements of the set of local subrings of K for the dominance relation.

(3) Let Γ be a totally ordered abelian group.

A valuation v of K with values in Γ is a map

$$v : K^* \longrightarrow \Gamma \quad \text{s.t.}$$

(a) $v(xy) = v(x) + v(y)$

(b) $v(x+y) \geqslant \min(v(x), v(y))$

The set of elements $x \in K^*$ s.t. $v(x) \geqslant 0 \in \Gamma$ $(\cup\{0\})$ is a valuation ring of K with maximal ideal

$$\{x \in K \mid x = 0 \text{ or } v(x) > 0\}$$

The valuation v is called discrete if $\Gamma = \mathbb{Z}$.

<u>The valuative criterion for separatedness:</u>

<u>Notation:</u> K a field, $R \subset K$ a valuation ring.

$\mathfrak{m} \subset R$ the maximal ideal of R

$T := \operatorname{Spec} R$ has exactly one closed point: $\mathfrak{m}$

$U := \operatorname{Spec} K$ one-pointed scheme

$R \hookrightarrow K \iff U = \operatorname{Spec} K \hookrightarrow \operatorname{Spec} R = T$

image is the generic point of T $\leftrightarrow (0) \subset R$ (see homework)

$(0) \subset \mathfrak{p} \;\; \forall \, \mathfrak{p} \in \operatorname{Spec} R \Rightarrow \overline{\{(0)\}} = T$

$\mathcal{O}_{T,\mathfrak{m}} = R_{\mathfrak{m}} = R \qquad \mathcal{O}_{T,(0)} = R_{(0)} = K$

<u>Theorem</u> (valuative criterion of separatedness)

Let $f: X \longrightarrow Y$ be a morphism of schemes. Assume X is noetherian. Then f is separated if and only if the following condition holds.

For any K and R as above, and any morphisms $T \to Y$, $U \to X$ forming the commutative diagram

$$\begin{tikzcd}
(0) \arrow[d, mapsto] & \operatorname{Spec} K = & U \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
(0) \in & \operatorname{Spec} R = & T \arrow[r] \arrow[ur, dotted, "i"] & Y
\end{tikzcd}$$

there is at most one morphism $i : T \to X$ making the whole diagram commutative.

For the proof, we need: (similar to homework)

<u>Lemma:</u> To give a morphism from $U = \operatorname{Spec} K$ to a scheme X is equivalent to giving a point $x_1 \in X$ and an inclusion of fields $k(x_1) \hookrightarrow K$. To give a morphism

from $T = \operatorname{Spec} R$ to X is equivalent to giving two points $x_0, x_1 \in X$, with $x_0 \in Z := \overline{\{x_1\}}$ and an inclusion of fields $k(x_1) \hookrightarrow K$ s.t. R dominates the local ring $\mathcal{O}_{Z, x_0}$, where Z is endowed with its reduced induced scheme structure.

<u>Proof</u>: The part about $U \to X$ was done in homework.

For the second part, let $t_0 \, (= m_R)$ be the closed point of T and $t_1 \, (= (0))$ the generic point of T.

Given a morphism $\varphi : T \to X$, let x_0 and x_1 be the images of t_0 and t_1. $Z := \overline{\{x_1\}}$

Since T is reduced, $T \xrightarrow{\varphi} X$ factors through X_{red}.

$$\begin{tikzcd} & T \arrow[rr, "\varphi"] \arrow[dr] & & X \\ Z \arrow[rr, hook] & & X_{red} \arrow[ur] & \end{tikzcd}$$

Claim: $T \xrightarrow{\varphi} X$ factors through Z.

as sets: $T \xrightarrow{\varphi} X$ factors through Z.

$$\begin{tikzcd} T \arrow[rr, "\varphi"] \arrow[dr, "\tilde{\varphi}"'] & & X \\ & Z \arrow[ur, hook, "j"'] & \end{tikzcd}$$

on the sheaves:

$$\begin{tikzcd} \mathcal{O}_X \arrow[rr, "\varphi^\#"] \arrow[dr, "j^\#"'] & & \varphi_* \mathcal{O}_T \\ & j_* \mathcal{O}_Z \arrow[ur, dotted, "?"'] & \end{tikzcd}$$

The factorization will exist when $\varphi^\#$ is zero on the kernel of $j^\#$: see exercise II.3.11

Assuming this, we have $T \longrightarrow Z \hookrightarrow X$

$\Rightarrow$ local hom. of local rings $\mathcal{O}_{Z,x_0} \longrightarrow \mathcal{O}_{T,t_0} = R_{\mathfrak{m}} = R$

$\Rightarrow$ $(R, \mathfrak{m})$ dominates $(\mathcal{O}_{Z,x_0}, \mathfrak{m}_{Z,x_0}) \subset K$

$\mathcal{O}_{Z,x_0} \hookrightarrow R \subset K$

$\mathfrak{m} \cap \mathcal{O}_{Z,x_0} \leftarrow \cdots \mathfrak{m}$