Remark: $\underbrace{\operatorname{Spec} R = T \hookrightarrow Z}\hookrightarrow X$

integral schemes: have generic points: t_1, x_1

$$\mathcal{O}_{Z,x_1} = \text{field} \supset \mathcal{O}_{Z,z} \quad \forall z \in Z$$

from the first part $\quad k(x_1) \subset K$

verify $\quad k(x_1) = \mathcal{O}_{Z,x_1} \cup \quad \cup$

$$\mathcal{O}_{Z,x_1} \supset \mathcal{O}_{Z,x_0}$$

verify $\quad \mathcal{O}_{Z,x_0} \subset R = \mathcal{O}_{T,t_0}$

R dominates $\mathcal{O}_{Z,x_0}$ means $\mathfrak{m}_R \cap \mathcal{O}_{Z,x_0} = \mathfrak{m}_{Z,x_0}$.

Proof of lemma continued.

Conversely, suppose we are given $x_0, x_1 \in X$, $x_0 \in Z = \overline{\{x_1\}}$ and the inclusion $k(x_1) \subset K$ s.t. R dominates $\mathcal{O}_{Z,x_0}$

The inclusion $\mathcal{O}_{Z,x_0} \hookrightarrow R$ gives a morphism

$T \longrightarrow \operatorname{Spec} \mathcal{O}_{Z,x_0}$ compose this with

$\operatorname{Spec} \mathcal{O}_{Z,x_0} \longrightarrow X$:

Given any affine neighborhood $U = \operatorname{Spec} A \subset X$ of x_0, then U also contains x_1 because $x_0 \in \overline{\{x_1\}}$

$\Leftrightarrow$ $x_1 \in$ any open set containing x_0

($x_1 \in X \setminus U \Rightarrow \overline{\{x_1\}} \subset X \setminus U \Rightarrow x_0 \in X \setminus U$)

$x_1 \in U \Leftarrow x_0 \in U$

We now have the evaluation map $A = \mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,x_0}$

$$\begin{tikzcd}
 & \operatorname{Spec} \mathcal{O}_{Z,x_0} \arrow[r, hook] \arrow[drr] & \operatorname{Spec} \mathcal{O}_{X,x_0} \arrow[r] & \operatorname{Spec} A = U \arrow[d, hook] & \mathcal{O}_{X,x_0} \arrow[d, two heads] \\
\Rightarrow & & & X & \mathcal{O}_{Z,x_0}
\end{tikzcd}$$

$\Rightarrow T \to \operatorname{Spec} \mathcal{O}_{Z,x_0} \longrightarrow \operatorname{Spec} A \to X$ $\square$

Some terminology: When $x_0 \in \overline{\{x_1\}}$, we say

- x_0 is a specialization of x_1
- x_1 is a generization of x_0

Proof of the valuative criterion of separatedness:

First suppose $f : X \longrightarrow Y$ is separated.

Suppose we have

$$\begin{tikzcd}
\operatorname{Spec} K = U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{Spec} R = T \arrow[r] \arrow[ur, dashed, "h"] \arrow[ur, dashed, bend right, "h'"'] & Y
\end{tikzcd}$$

and two morphisms h, h' making the diagram commutative. We show $h = h'$.

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[drrr, bend left=40, "h|_U"] \arrow[drr, "(h,h')|_U"'] \arrow[ddrr, bend right=40, "h'|_U"'] & T \arrow[dr, "(h,h')"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "h'"'] & & \\
& & X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
& & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

$h|_U = h'|_U$

<u>Claim</u>: (exercise. follows from the universal property of fiber products)

a morphism $g : W \to X \times_Y X$ $\quad$ W any scheme

factors through $\Delta : X \to X \times_Y X$ iff $p_1 \circ g = p_2 \circ g$

$h|_U = h'|_U \Rightarrow$ $\quad (h, h')|_U : U \to X \times_Y X$ factors

through Δ:

$$\begin{tikzcd} (h,h')|_U : U \arrow[r] \arrow[d, hook] & X \arrow[r, "\Delta"] & X \times_Y X \\ T & & \end{tikzcd}$$

image is the generic point of T

Δ is a closed embedding and $\Delta(X)$ contains the image of the generic point of $T \Rightarrow \Delta(X) \supset$ image of T

$\Rightarrow \quad \Delta(x) \ni$ image of the closed point of $T = \operatorname{Spec} R$

Claim (from previous page) $\Rightarrow$ $h(t_0) = h'(t_0) =: x_0$

$x_1 := h(t_1) = h'(t_1) =$ image of $U = \operatorname{Spec} K$

$\Rightarrow \quad x_0 \in \overline{\{x_1\}} =: Z$ with reduced induced scheme structure

and $\quad k(x_1) = \mathcal{O}_{Z,x_1} \subset K$ given by $h|_U = h'|_U$

$U = \operatorname{Spec} K \longrightarrow X$

$x_0 = h(t_0) = h'(t_0) \Rightarrow \quad \mathcal{O}_{Z,x_0} \subset R$ dominated.

previous lemma $\Rightarrow \quad h = h' : \operatorname{Spec} R \longrightarrow X$

Conversely, suppose the valuative criterion is satisfied.

We show that $\Delta(X) \subset X \times_Y X$ is closed.

We use the following lemma:

<u>Lemma:</u> Let $f: X \to Y$ be a quasi-compact morphism of schemes. The subset $f(X)$ of Y is closed iff it is closed under specialization, meaning $\forall y \in f(X)$, any specialization of y also belongs to $f(X)$.

<u>Proof:</u> Hartshorne

<u>Remark:</u> In general any closed subset is closed under specialization.

Since X is noetherian, $\Delta: X \longrightarrow X \times_Y X$ is quasi-compact.

We show $\Delta(X)$ is closed under specialization.

Choose $\xi_1 \in \Delta(X) \subset X \times_Y X$ and $\xi_0 \in Z := \overline{\{\xi_1\}} \subset X \times_Y X$

we show $\xi_0 \in \Delta(X)$.

endow Z with the reduced induced scheme structure. ξ_1 is the generic point of Z

$$\Rightarrow \quad \mathcal{O}_{Z,\xi_1} =: K \text{ is a field}$$
$$= k(\xi_1) \text{ residue field of } \xi_1 \text{ in } X \times_Y X$$

$$\mathcal{O}_{Z,\xi_0} \subset K$$

$\exists$ valuation ring $R \subset K$ which dominates $\mathcal{O}_{Z,\xi_0}$.

Using the lemma from the previous lecture, we obtain a morphism $g: T := \operatorname{Spec} R \longrightarrow X \times_Y X$

s.t. $g(t_1) = \xi_1$, $g(t_0) = \xi_0$

$h := p_1 \circ g$ $\qquad h' := p_2 \circ g$ $\qquad h|_U = h'|_U$ $\qquad U = \operatorname{Spec} K$

because $\xi_1 = g(t_1) \in \Delta(X)$
(see Claim about Δ)

So $(p_1 \circ g)|_U = (p_2 \circ g)|_U \Longrightarrow p_1 \circ g = p_2 \circ g$

valuative criterion

(use claim Δ again) $\Rightarrow$ g factors through Δ.

$\Rightarrow \xi_0 = g(t_0) \in \Delta(X)$. $\square$

Recall that properness is a substitute for compactness.
In a compact top. space, any closed subset is also compact, so any continuous map from a compact topological space is closed.
We use the condition of universally closed to define properness: being closed is not enough: $\mathbb{A}^1 \longrightarrow \operatorname{Spec} k$ is closed but it is not proper.

$$\begin{array}{c} \mathbb{A}^1 \\ \cap \\ \mathbb{P}^1 \end{array}$$

Universally closed uses base change:

base change replaces the notion of extension of scalars:

V vector space over $\mathbb{Q}$: $\rightsquigarrow$ $V \underset{\mathbb{Q}}{\otimes} \mathbb{R}$ vector space$/\mathbb{R}$

Recall that tensor product geometrically correspond to fiber products.

We use fiber products to define base change.