We need to define fractions more generally:

Idea: The construction of $\mathbb{Q}$:

$$\mathbb{Q} := \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) / \sim$$

$$(a, b) \sim (c, d) \iff ad - bc = 0$$

$$\frac{a}{b} = \frac{c}{d}$$

Def: For a ring R, a subset $S \subset R$ is called a multiplicative set if (1) $1 \in S$

(2) $\forall\, s, t \in S$, $st \in S$

Def: Given $S \subset R$ multiplicative, define an equivalence relation $\sim$ on $R \times S$ as $(a, s) \sim (b, t) \iff \exists\, s' \in S$

$$(at - bs)s' = 0$$

The localization $S^{-1}R$ of R at S is

$$S^{-1}R := R \times S / \sim$$

Notation: The equivalence class of (a,s) is denoted $\frac{a}{s}$

We have the natural map $R \xrightarrow{can} S^{-1}R$

$$r \mapsto \frac{r}{1}$$

Note: If $\exists$ $t \in R$ and $s \in S$ s.t. $t \neq 0$ and $ts = 0$,

then $\frac{t}{1} = \frac{0}{1}$ in $S^{-1}R$ because $(t \cdot 1 - 0 \cdot 1)s = 0$

Localizations have a universal property:

Prop.: Suppose $S \subset R$ is a multiplicative set and $\varphi : R \to B$ is a morphism of rings s.t. $\forall s \in S$, $\varphi(s)$ is invertible in B. Then $\exists$ a unique homomorphism of rings $S^{-1}\varphi : S^{-1}R \longrightarrow B$ s.t.

$$S^{-1}\varphi\left(\frac{a}{s}\right) = \varphi(a)\varphi(s)^{-1}$$

$\forall\ a \in R,\ s \in S$

i.e.

$$\begin{tikzcd} R \arrow[rr, "\varphi"] \arrow[dr, "\text{can.}"'] \arrow[r, phantom, shift right=4ex, "\circlearrowleft"] & & B \arrow[dl, "S^{-1}\varphi"] \\ & S^{-1}R & \end{tikzcd}$$

Atiyah-McDonald, Chapter 3.

Note: $\forall\ s \in S$ $\quad \frac{s}{1} \in S^{-1}R$ is invertible:

$$\frac{s}{1} \cdot \frac{1}{s} = \frac{1}{1} \quad \text{because:}$$

$$\frac{s \cdot 1}{1 \cdot s} = \frac{1}{1} \quad \text{because} \quad s \cdot 1 \cdot 1 - 1 \cdot s \cdot 1 = 0$$

Note: If $0 \in S$, then $S^{-1}R = \{0\}$

Main examples of multiplicative sets for us:

(1) $S = \{1, f, f^2, f^3, \dots, f^n, \dots\}$ for some $f \in R$

then $S^{-1}R = R[f^{-1}]$

(2) $S = R \setminus \mathfrak{p}$ for $\mathfrak{p} \subset R$ prime ideal

We denote $S^{-1}R = R_{\mathfrak{p}}$ in this case.

<u>Lemma:</u> $\forall$ R, $\mathfrak{p} \in \operatorname{Spec} R$, the ring $R_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$ (:= the ideal in $R_{\mathfrak{p}}$ generated by the image of $\mathfrak{p}$ under can)

<u>Idea of proof:</u> use the fact that a ring R is local with maximal ideal $\mathfrak{m}$ $\Leftrightarrow$ (x invertible $\Leftrightarrow$ $x \notin \mathfrak{m}$)

Proposition: $\forall$ ring R, $\operatorname{Spec} R$ is a locally ringed space.

In fact, $\forall\, \mathfrak{p} \in \operatorname{Spec} R$, the stalk $\mathcal{O}_{\mathfrak{p}} := \mathcal{O}_{\operatorname{Spec} R, \mathfrak{p}}$

is naturally isomorphic to $R_{\mathfrak{p}}$.

The proposition will follow from these lemmas:

Lemma 1: $\forall$ sheaf $\mathcal{F}$ on X, for any basis $\{V_i, i \in I\}$

of the topology of X, $\forall\, x \in X$, we have

$$\mathcal{F}_x \cong \varinjlim_{\substack{i \in I \\ x \in V_i}} \mathcal{F}(V_i)$$

Lemma 2: $\forall$ ring R, $\forall\, \mathfrak{p} \in \operatorname{Spec} R$, $\varinjlim_{f \notin \mathfrak{p}} R[f^{-1}] \cong R_{\mathfrak{p}}$

First: Lemma 1 and Lemma 2 $\Rightarrow$ Proposition!

By the commutative algebra result, we only need to check $\mathcal{O}_p \cong R_p$

Take the basis of the topology of Spec$R$ formed by the basic open sets U_f, for $f \in R$

$$\mathcal{O}_p \underset{\text{lemma 1}}{=\!=} \varinjlim_{p \in U_f} \mathcal{O}(U_f) = \varinjlim_{f \notin p} \mathcal{O}(U_f)$$

$$= \varinjlim_{f \notin p} R[f^{-1}] \underset{\text{lemma 2}}{=\!=\!=} R_p$$

Proof of Lemma 1:

$$\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U) \overset{?}{=} \varinjlim_{x \in U_i} \mathcal{F}(U_i)$$

we have the natural

$$\begin{tikzcd}
\coprod\limits_{\substack{i \in I \\ x \in U_i}} \mathcal{F}(U_i) \arrow[r, hook] \arrow[dr, "\rho", "\text{composition}"'] & \coprod\limits_{x \in U} \mathcal{F}(U) \arrow[d, two heads] \\
& \mathcal{F}_x
\end{tikzcd}$$

We show that the map ρ factors through a bijection $\varinjlim_{x \in U_i} \mathcal{F}(U_i) \longrightarrow \mathcal{F}_x$

(a) factorization: if $(U_i, s_i) \sim (U_j, s_j)$, then $s_i(x) = s_j(x)$

$$(U_i, s_i) \sim (U_j, s_j) \iff \exists\, k \text{ s.t. } U_k \subset U_i \cap U_j$$
$$\text{and } s_i|_{U_k} = s_j|_{U_k}$$
$$\Rightarrow \quad s_i(x) = s_j(x)$$

(a) ρ is injective: if (U_i, s_i) and (U_j, s_j) have the same image in $\mathcal{F}_x$, then $\exists\, U \subset U_i \cap U_j$, $U \ni x$

s.t. $s_i|_U = s_j|_U$

because $\{U_i, i \in I\}$ is a basis of the topology,

$\exists\, k \in I$ s.t. $x \in U_k \subset U$

then $s_i|_{U_k} = s_j|_{U_k}$ $\Rightarrow$ (U_i, s_i) and (U_j, s_j) represent the same element of $\varinjlim_{\substack{i \in I \\ x \in U_i}} \mathcal{F}(U_i)$

(b) ρ is surjective: Given a pair (U, s) representing an element of $\mathcal{F}_x$, $\exists\, i \in I$ s.t. $x \in U_i \subset U$

then $(V, s) \sim (V_i, s|_{V_i})$ is in the image of ρ. $\square$

<u>Proof of Lemma 2:</u> Given $\mathfrak{p} \in \operatorname{Spec} R$, for any $f \notin \mathfrak{p}$ we have the natural map, obtained from the universal property of localizations:

$$\begin{tikzcd} f \in R \arrow[r, "\varphi"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & R_{\mathfrak{p}} \\ \frac{f}{1} \in R[f^{-1}] \arrow[ur, "S^{-1}\varphi"'] & \end{tikzcd}$$

$\Rightarrow \frac{f}{1} = \varphi(f)$

$\varphi(f)$ is invertible $\Rightarrow$

$S = \{1, f, f^2, \ldots\}$

$$\Rightarrow \coprod_{f \notin \mathfrak{p}} R[f^{-1}] \longrightarrow R_{\mathfrak{p}}$$

We show this induces an isomorphism $\varinjlim_{f \notin \mathfrak{p}} R[f^{-1}] \xrightarrow{\cong} R_{\mathfrak{p}}$

(a) We determine when two elements of $\coprod_{f \notin \mathfrak{p}} R[f^{-1}]$ have the same image in $R_{\mathfrak{p}}$.

Suppose given $f, g \notin \mathfrak{p}$ $\quad \frac{a}{f^m} \in R[f^{-1}]$, $\frac{b}{g^n} \in R[g^{-1}]$

when do we have $\quad \frac{a}{f^m} = \frac{b}{g^n}$ in $R_{\mathfrak{p}}$

this happens when $\exists\ s \notin \mathfrak{p}$ s.t. $s(a \cdot g^n - b f^m) = 0$

Put $h := fgs$, then $\frac{a}{f^m} = \frac{b}{g^n}$ in $R[h^{-1}]$

$$R[h^{-1}] = R[f^{-1}][g^{-1}][s^{-1}]$$

So $\frac{a}{f^m}$ and $\frac{b}{g^n}$ represent the same element of $\varinjlim_{t \notin \mathfrak{p}} R[t^{-1}]$