The Proj of a graded ring

Def: (e.g. Atyiah–McDonald)

A graded ring S is a commutative ring with 1 and a direct sum decomposition (as abelian groups)

$$S = \bigoplus_{\substack{d \in \mathbb{Z} \\ d \geq 0}} S_d$$

s.t. $S_d \cdot S_e \subset S_{d+e} \quad \forall\ d, e$

Note: $S_0 \subset S$ is a subring

and $S_+ := \bigoplus_{d > 0} S_d$ is an ideal

Def (Proj S) As a set:

$$\operatorname{Proj} S := \left\{ \mathfrak{p} \,\middle|\, \begin{array}{l} \mathfrak{p} \subset S \text{ homogeneous} \\ \text{prime ideal} \\ \mathfrak{p} \not\supset S_+ \end{array} \right\}.$$

Def: an element f of S is homogeneous of degree d if $f \in S_d \setminus \{0\}$.

Def: An ideal $I \subset S$ is homogeneous if it can be generated by homogeneous elements.

Notation: for $I \subset S$, we write $I_d := I \cap S_d$

Lemma: $I \subset S$ ideal is homogeneous iff $I = \bigoplus_{d \geq 0} I_d$

Def: (Topology of Proj S)

The closed sets of Proj S are

$$Z(I) := \{ \mathfrak{p} \in \operatorname{Proj} S \mid \mathfrak{p} \supset I \} \subset \operatorname{Proj} S$$

for all $I \subset S$ homogeneous ideal.

As in the affine case, we have basic open sets:

For all $f \in S_d$ for $d \in \mathbb{Z}_{\geq 0}$, we define

$$U_f := \{ \mathfrak{p} \in \operatorname{Proj} S \mid f \notin \mathfrak{p} \}$$

Def: (Sheaf of rings) $f \in S$ homogeneous:

$$\mathcal{O}_{\operatorname{Proj} S}(U_f) := S[f^{-1}]_0$$ the subring of degree 0 elements in $S[f^{-1}]$

For arbitrary open sets U:

$$\mathcal{O}_{\mathrm{Proj}\,S}(U) = \varprojlim_{U_f \subset U} \mathcal{O}_{\mathrm{Proj}\,S}(U_f) = \varprojlim_{U_f \subset U} S[f^{-1}]_0$$

__Note:__ When we dehomogenize with respect to x_i:

the ring of V_i is $k\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right]$

$$= k[x_0, \dots, x_n][x_i^{-1}]_0$$

__Lemma:__ $\forall$ $p \in \mathrm{Proj}\,S$, we have a canonical isomorphism $\mathcal{O}_{\mathrm{Proj}\,S,\, p} \cong (S_p)_0$ the subring of elements of degree 0 in S_p.

Proof: Use the result in the affine case together with the fact that taking subrings of degree 0 elements commutes with localization (e.g. use the universal property of localizations to prove this) $\square$

Corollary: $(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S})$ is a locally ringed space.

Recall that $\mathbb{P}^n_k = \bigcup_{i=0}^{n} U_i$ where $U_i \cong \mathbb{A}^n_k$

This can be generalized: First recall that any open subset of a locally ringed space inherits a structure of locally ringed space. (If $V \subset U \subset X$, $\mathcal{O}_U(V) := \mathcal{O}_X(V)$)

So each basic open set $U_f \subset \mathrm{Proj}\, S$ inherits a structure of locally ringed space.

Lemma: With this structure: $U_f \cong \mathrm{Spec}\, S[f^{-1}]_0$

isom. of loc. ringed spaces.

Proof: We have a morphism

$$\varphi : U_f \longrightarrow \mathrm{Spec}\, S[f^{-1}]_0$$

obtained from the hom. of rings

$$S[f^{-1}]_0 = \mathcal{O}(U_f) \xleftarrow{\ \mathrm{Id}\ } S[f^{-1}]_0$$

We need to show that φ is an isomorphism.

First: φ is a bijection on the underlying sets.

Recall how φ is defined:

$\sigma \in U_f$

$(f \notin \sigma)$

$\Downarrow$

$S[f^{-1}]_\sigma = S_\sigma$

$$\begin{array}{ccc} S[f^{-1}]_0 & \xrightarrow{\alpha = Id} & \mathcal{O}_{\mathrm{Proj}\, S}(U_f) = S[f^{-1}]_0 \\ & \searrow & \downarrow ev_\sigma \\ & & \mathcal{O}_{\mathrm{Proj}\, S, \sigma} \cong (S_\sigma)_0 \\ & & \cup \\ & & m_\sigma = (\sigma S_\sigma)_0 \end{array}$$

$$\varphi(\sigma) = (\sigma S[f^{-1}])_0$$

Injectivity means: $\forall\ \sigma_1, \sigma_2 \in U_f$

$$(\sigma_1 S[f^{-1}])_0 = (\sigma_2 S[f^{-1}])_0 \implies \sigma_1 = \sigma_2$$

Surjectivity means: $\forall\ \sigma_0 \subset (S[f^{-1}])_0$ prime, $\exists$

$\sigma \in U_f$ s.t. $\sigma_0 = (\sigma S[f^{-1}])_0$

Reference: Vakil's notes ("The rising sea") p. 149. □

Def: For any ring R, we define projective n-space

$$\mathbb{P}^n_R := \operatorname{Proj} R[x_0, \dots, x_n]$$

We can generalize projective algebraic sets:

$Y = Z(I)$ $\quad I$ homogeneous ideal

$S(Y) := S/I(Y)$ $\quad S = R[x_0, \dots, x_n]$

$Y = \operatorname{Proj} S(Y)$. (= def. of $Z(I)$ as a scheme)

Corollary of Lemma: $\operatorname{Proj} S$ is a scheme.

First properties of schemes:

Def: (1) A scheme is connected / irreducible / quasi-compact if its underlying topological space is connected / irreducible / quasi-compact

(2) A scheme is reduced if $\forall$ open sets $U \subset X$, the ring $\mathcal{O}_X(U)$ is reduced, i.e., $\mathcal{O}_X(U)$ has no nilpotents (i.e., $\forall\, a \in \mathcal{O}_X(U)$, $a^n = 0 \Rightarrow a = 0$ $\forall n$)

Homework: Equivalently, $\forall\, x \in X$, $\mathcal{O}_{X,x}$ is reduced.

(3) A scheme is integral if, for all nonempty open sets $U \subset X$, $\mathcal{O}_X(U)$ is an integral domain.

($\{0\}$ is <u>not</u> an integral domain)

(4) A scheme is locally noetherian if it can be covered by affine open sets $U \cong \operatorname{Spec} A$ s.t. A is noetherian ring.

(5) A scheme is noetherian if it is locally noetherian and quasi-compact

Exercise: Equivalently, it has a finite cover by $U \cong \operatorname{Spec} A$ with A noetherian.

Prop.: A scheme is integral iff it is irreducible and reduced.

Proof: First suppose X is integral.
Clearly, X is reduced. We show X is irreducible.

If $X = X_1 \cup X_2$ $\quad X_1, X_2$ closed

put $U_i := X \setminus X_i$

$X = X_1 \cup X_2 \Rightarrow U_1 \cap U_2 = \emptyset$

So $U_1 \cup U_2 = U_1 \amalg U_2$

Claim: $\mathcal{O}(U_1 \amalg U_2) \longrightarrow \mathcal{O}(U_1) \times \mathcal{O}(U_2)$

$s \longmapsto (s|_{U_1}, s|_{U_2})$

is an isom. by the sheaf properties.

$\mathcal{O}(U_1 \amalg U_2) \cong \mathcal{O}(U_1) \times \mathcal{O}(U_2)$ is an integral domain. This is only possible if $\mathcal{O}(U_1) = \{0\}$ or $\mathcal{O}(U_2) = \{0\}$ $\Longrightarrow U_1 = \emptyset$ or $U_2 = \emptyset$ because $\{0\}$ is not an integral domain.

Now suppose X is irreducible and reduced.

Let $U \subset X$ be open and not empty.

Let $f, g \in \mathcal{O}_X(U)$ s.t. $fg = 0$

We saw in the big proof in the previous lecture, that $\forall$ locally ringed space, $\forall f \in \mathcal{O}_X(X)$, the set $X_f := \{x \in X \mid f(x) \notin m_x \subset \mathcal{O}_{X,x}\}$ is open.

The set of zeros of f is $Z(f) := \{x \in X \mid f(x) \in m_x\}$ which is closed.