First properties of morphisms of schemes:

<u>Definition:</u> (1) A morphism of schemes $f: X \to Y$ is locally of finite type if $\exists$ a covering of Y by open affine subsets $V_i = \operatorname{Spec} B_i$ s.t. $\forall i$ $f^{-1}(V_i)$ has a covering by open affine subsets $U_{ij} := \operatorname{Spec} A_{ij}$ where $\forall i, j$

A_{ij} is a finitely generated B_i-algebra.

(a small explanation: $f: X \to Y$ $\quad f^{\#}: \mathcal{O}_Y \to f_* \mathcal{O}_X$

$\forall i, f^{-1}(V_i) = \bigcup_j U_{ij} \longrightarrow V_i$ or $\forall i, j, f|_{U_{ij}}: U_{ij} \longrightarrow V_i$

$$f|_{U_{ij}} : \operatorname{Spec} A_{ij} \longrightarrow \operatorname{Spec} B_i$$

$$A_{ij} \longleftarrow B_i$$

gives A_{ij} a structure of B_i-algebra.)

(2) A morphism $f: X \to Y$ is of finite type if it is locally of finite type and, with the above notation, $\forall i$ the cover $\{U_{ij}\}$ of $f^{-1}(V_i)$ is finite.

(3) A morphism of schemes $f: X \to Y$ is finite if $\exists$ a covering $Y = \bigcup_{i \in I} V_i$ with $V_i = \operatorname{Spec} B_i$ open affine s.t. $\forall i$ $f^{-1}(V_i)$ is affine $= \operatorname{Spec} A_i$ with A_i a finite B_i-algebra (i.e., A_i is a finitely generated B_i-module).

(4) An open subscheme of a scheme X is an open subset $U \subset X$ with topology induced from X and sheaf of rings $\mathcal{O}_U := \mathcal{O}_X|_U$, meaning $\forall V \subset U$, $\mathcal{O}_U(V) = \mathcal{O}_X(V)$.

(5) An open embedding is a morphism of schemes $f: X \longrightarrow Y$ s.t. $f: X \hookrightarrow Y$ embeds X as an open subset of Y and the sheaf of rings on X is isomorphic to that induced by Y on the image of X.

(6) A closed embedding is a morphism of schemes $f: X \longrightarrow Y$ which induces a homeomorphism of X with a closed subset of Y and such that the morphism of sheaves $f^{\#}: \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X$ is surjective. The kernel of $f^{\#}$ is then the sheaf of ideals of (the image of) X in Y.

(7) A closed subscheme of a scheme Y is a closed subset $i: X \hookrightarrow Y$ with a sheaf of rings $\mathcal{O}_X$, s.t. $(X, \mathcal{O}_X)$ is a scheme and $\exists$ surjective homomorphism

$$i^{\#}: \mathcal{O}_Y \twoheadrightarrow i_* \mathcal{O}_X.$$

The sheaf of ideals of X is $(\ker i^{\#}) \subset \mathcal{O}_Y$.

In other words, a closed subscheme $X \hookrightarrow Y$ is an equivalence class of closed embeddings $j: Z \hookrightarrow Y$, where two closed embeddings $j: Z \hookrightarrow Y$, $j': Z' \hookrightarrow Y$ are equivalent if $\exists$ an isomorphism $\varphi: Z \xrightarrow{\sim} Z'$ s.t. $j' \circ \varphi = j$

$$\begin{tikzcd} Z \arrow[rr, "\sim"] \arrow[dr, "j"'] & & Z' \arrow[dl, "j'"] \\ & Y \arrow[u, phantom, "\cong"] & \end{tikzcd}$$

You will see in homework that for an affine scheme $\operatorname{Spec} A$, any closed subscheme is of the form $\operatorname{Spec} A/I$ for some ideal I.

Some examples: (1) Affine varieties: $Y \subset \mathbb{A}^n = \operatorname{Spec} A$

$A := k[y_1, \ldots, y_n]$ $\quad I(Y) \quad A(Y) = A/I(Y)$

$Y = \operatorname{Spec} A(Y) \hookrightarrow \mathbb{A}^n$ closed embedding

(2) Projective varieties: $Y \hookrightarrow \mathbb{P}^n = \operatorname{Proj} S$

$S := k[x_0, \ldots, x_n]$ $\quad I(Y) \subset S, \ S(Y) = S/I(Y)$

$Y = \operatorname{Proj} S(Y) \hookrightarrow \mathbb{P}^n$ closed embedding

(3) Quasi-affine varieties are open subschemes of affine varieties.

(4) Quasi-projective varieties are open subschemes of projective varieties.

(5) Finite morphisms: A integral domain

$$A \subset K = \mathrm{Frac}(A).$$

$B :=$ the integral closure of A in K

$:=$ the set of elements of K that are integral $/A$

$:=$ the set of elements of K which satisfy monic polynomial equations with coefficients in A.

$$:= \{x \in K \mid \exists \ a_1, \dots, a_n \in A \text{ with } x^n + a_1 x^{n-1} + \cdots + a_n = 0\}$$

Atiyah-McDonald (Integral dependence and valuations): this is a subring of K.

B is a finitely generated A-module, i.e., a finite A-algebra. (need B to be a finitely generated A-alg.)

$$A \subset B \subset K \Rightarrow \operatorname{Spec} B \to \operatorname{Spec} A$$ finite morphism of schemes.

$\operatorname{Spec} B$ is the "normalization" of $\operatorname{Spec} A$.

<u>Example:</u> $A := k[x,y]/(x^3 - y^2) = A(Y)$ Y is a cuspidal cubic

$Y = Z(x^3 - y^2) \subset \mathbb{A}^2_k$

A is an integral domain.

Claim $K := K(A) = \operatorname{Frac}(A) \trianglelefteq k(t)$

send $A \hookrightarrow k(t) \supset k[t]$ integrally closed

$x \longmapsto t^2$

$y \longmapsto t^3$

We have $\quad A \subset k[t] \subset k(t)$

<u>Claim</u> $\quad k[t]$ is the integral closure of A in $k(t)$.

<u>Proof</u>: We already know, the integral closure of A is $\subset k[t]$ because $k[t]$ is integrally closed.

We need to show that the elements of $k[t]$ satisfy monic polynomials over A. Only need to prove it for a generating set, e.g. $\{t\}$: t satisfies $X^2 - x = 0$

or $\quad X^3 - y = 0$

$$A = k[x,y]/(x^3 - y^2) \hookrightarrow k[t]$$

$$\operatorname{Spec} A = Y \longleftarrow \operatorname{Spec} k[t] = \mathbb{A}^1_k$$

(b) $\quad X$ any scheme, $f \in \mathcal{O}_X(X)$, $\quad X_f \underset{\text{open}}{\subset} X$

$$X_f = \{x \in X \mid f(x) \notin \mathfrak{m}_x \subset \mathcal{O}_{X,x}\}$$

$$(X_f, \mathcal{O}_X|_{X_f}) \hookrightarrow (X, \mathcal{O}_X)$$

open embedding.

(7) A any ring, $\operatorname{Spec} A$

$\cup$

I any ideal $A \twoheadrightarrow A/I$

gives a closed embedding $\operatorname{Spec}(A/I) \hookrightarrow \operatorname{Spec} A$

We think of this as the closed subscheme of $\operatorname{Spec} A$ defined by the ideal I. The morphism of sheaves is surjective because it is surjective on the stalks.

Note that I is arbitrary and need not be radical.

<u>example</u>: in $\mathbb{A}^2 = \operatorname{Spec} k[x,y]$

$Y = Z(x)$: $I(Y) = (x)$ $A(Y) = k[x,y]/(x) \cong k[y]$

the y-axis.

$Y' = Z(x^2) = \operatorname{Spec} k[x,y]/(x^2)$ $\quad A(Y') = k[x,y]/(x^2)$

$$A(Y') \twoheadrightarrow A(Y)$$

$$k[x,y]/(x^2) \twoheadrightarrow k[x,y]/(x) \cong k[y]$$

$$\mathbb{A}^2 \hookleftarrow Y' \hookleftarrow Y$$

$=$ as sets $=$ y-axis.

Y is reduced, Y' is not reduced: x is a nilpotent in $A(Y')$

Non-reduced schemes naturally occur as "limits" of reduced schemes: e.g.: $Z(x^2 - ty^2) \subset \mathbb{A}^2$ $\quad t \in k$.

If $t \neq 0$, $(x^2 - ty^2)$ is a radical ideal

If $t = 0$ $(x^2 - ty^2) = (x^2)$ is <u>not</u> radical

$Z(x^2)$ is the "limit" of $Z(x^2 - ty^2)$ as $t \to 0$.

Given a closed subset Y of a scheme X, there are many closed subschemes of X supported on Y, i.e., their underlying topological space is Y. The set of closed subschemes of X, supported on Y has a minimal element (i.e., it is a closed subscheme of all the other closed subschemes supported on Y). We call the scheme structure on Y given by this minimal element the reduced induced scheme structure on Y. It is defined as follows.

<u>Def</u>: Given $Y \subset X$ closed subset, the reduced induced scheme structure on Y is defined as follows.

For any open affine $U \subset X$, $U = \operatorname{Spec} A$, let the ideal of $Y \cap U$ be

$$I_{red} := \bigcap_{p \in U \cap Y} p \qquad (p \subset A \text{ prime}).$$

In other words, if $Y \cap U = V(I)$ for some ideal $I \subset A$, then $I_{red} = \bigcap_{p \in U \cap Y} p = \bigcap_{p \in V(I)} p = \bigcap_{p \supset I} p = \sqrt{I}$.

<u>Note:</u> If $I_1 \subset I_2$, then $A/I_1 \twoheadrightarrow A/I_2$ and $\operatorname{Spec} A/I_2 \underset{\text{closed subscheme}}{\hookrightarrow} \operatorname{Spec} A/I_1$ (details: exercise)

$\Rightarrow$ If $Y \cap U = V(I)$, then $\operatorname{Spec} A/\sqrt{I} \underset{\text{closed subscheme}}{\hookrightarrow} \operatorname{Spec} A/I$.