$$k[x,y,t]/(xy-t) \otimes_{k[t]} k[t]/(t-a)$$

$$0 \to (t-a) \to k[t] \to k[t]/(t-a) \to 0$$

tensor:

$$k[x,y,t]/(xy-t) \otimes_{k[t]} (t-a) \to \underbrace{k[x,y,t]/(xy-t) \otimes_{k[t]} k[t]}_{\cong\, k[x,y,t]/(xy-t)} \to$$

$$\to k[x,y,t]/(xy-t) \otimes_{k[t]} k[t]/(t-a) \to 0$$

$$\Rightarrow \quad k[x,y,t]/(xy-t) \otimes_{k[t]} k(y) \cong k[x,y,t]/(xy-t,\, t-a)$$

$$\cong k[x,y]/(xy-a)$$

$$\Rightarrow \quad X_y = \operatorname{Spec} k[x,y]/(xy-a) \quad \text{at } y = (t-a) \text{ "="} a \in k$$

We have a family of hyperbolas X_y parametrized by $\mathbb{A}^1$, embedded in $\mathbb{A}^3$. When $a \neq 0$, $(xy-a)$ is prime $\Rightarrow$ X_y is irreducible (it is also reduced)

When $a=0$, then X_y is reducible, ideal is (xy)

$$X_y = \operatorname{Spec} k[x] \cup \operatorname{Spec} k[y] \subset \mathbb{A}^2 = \operatorname{Spec} k[x,y]$$

meet at one point.

$$= \operatorname{Spec} k[x,y]/(xy)$$

Similarly, when $X = \operatorname{Spec} k[x,y,t]/(ty-x^2) \subset \mathbb{A}^3_k$

we obtain a family of parabolas: $X_y = \operatorname{Spec} k[x,y]/(ay-x^2)$

When $a \neq 0$, X_y is an integral scheme.

When $a = 0$, $X_y = \operatorname{Spec} k[x]/(x^2)$ is non reduced.

- Separated and proper morphisms.

The Zariski topology is almost never Hausdorff.
(e.g. $\mathbb{A}^1_k$, k alg. closed).

Separated $\leftrightarrow$ Hausdorff

Proper $\leftrightarrow$ proper topologically

Idea: when a topology is Hausdorff, the diagonal is closed in the product topology on $X \times X$.

Def: Given $X \xrightarrow[\pi_X]{} S$, the diagonal morphism of X over S is the unique morphism

$\Delta : X \longrightarrow X \times_S X$ s.t. $p_1 \circ \Delta = p_2 \circ \Delta = \mathrm{Id}_X$,

i.e.,

$$
\begin{tikzcd}
X \arrow[dr, "\exists!\Delta"] \arrow[drr, bend left, "\mathrm{Id}_X"] \arrow[ddr, bend right, "\mathrm{Id}_X"'] & & \\
& X \times_S X \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "\pi_X"] \\
& X \arrow[r, "\pi_X"'] & S
\end{tikzcd}
$$

<u>Definition</u>: A morphism $f : X \to Y$ is separated if the diagonal $\Delta : X \longrightarrow X \times_Y X$ is a closed embedding.

<u>Lemma</u>: Any morphism of affine schemes is separated.

<u>Proof:</u> $f : X \longrightarrow Y \qquad X = \operatorname{Spec} A, \ Y = \operatorname{Spec} B$

$$\Longleftrightarrow \qquad f^\# : B \longrightarrow A$$

$$X \times_Y X = \operatorname{Spec} A \otimes_B A$$

$$\begin{tikzcd}
A & & \\
& A \otimes_B A \arrow[ul, "\Delta^\#"'] & A \arrow[l, "a \otimes 1 \leftarrow\!\shortmid\, a"'] \arrow[ull, bend right, "\mathrm{Id}"'] \\
& A \arrow[u, "1 \otimes a \,\leftarrow\!\shortmid\, a \in"] \arrow[uul, "\mathrm{Id}"] \arrow[ur, phantom, "\sqsupset"] & B \arrow[l, "f^\#"] \arrow[u, "f^\#"']
\end{tikzcd}$$

$$\Delta^\#(1 \otimes a) = \Delta^\#(a \otimes 1) = a$$
$$\forall a \in A$$

(uniqueness of $\Delta^\#$) $\Longrightarrow$

$$\Delta^\#(a \otimes a') = a a'$$

$$\Delta : \operatorname{Spec} A \longrightarrow \operatorname{Spec} A \otimes_B A$$

Δ is a closed embedding because $A \otimes_B A \longrightarrow\!\!\!\!\!\rightarrow A$ is surjective $\qquad \square$

Note: the ideal of Δ in the affine case is

$$\mathrm{Ker}\left(A \otimes_B A \twoheadrightarrow A \right)$$

Quintessential example of a non-separated morphism:

k a field

$$\begin{array}{ccc} X = \mathrm{Spec}\, k[x] \, , & Y = \mathrm{Spec}\, k[y] & k[y] \\ \downarrow & \downarrow & \uparrow \\ \mathrm{Spec}\, k & \mathrm{Spec}\, k & k \end{array}$$

Glue X to Y as follows:

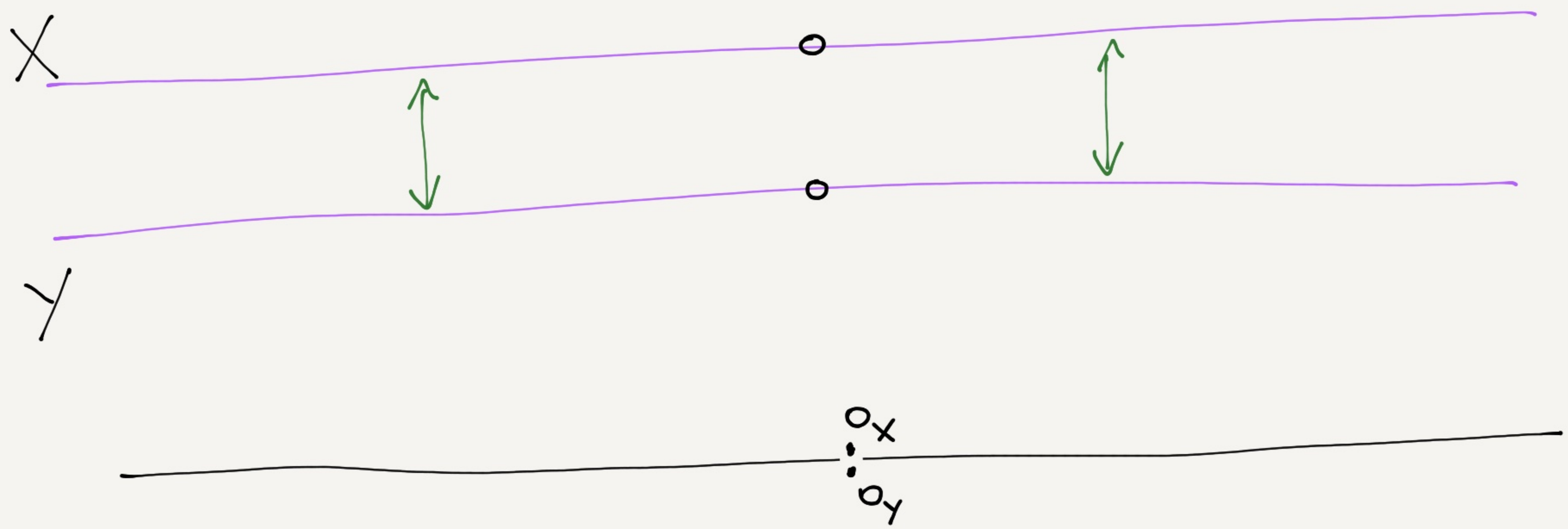
$U \subset X \quad : \quad U = \mathrm{Spec}\, k[x, x^{-1}]$

$V \subset Y \quad : \quad V = \mathrm{Spec}\, k[y, y^{-1}]$

$$\varphi : U \xrightarrow{\cong} V \, , \quad \varphi^{\#} : k[x, x^{-1}] \xleftarrow{\cong} k[y, y^{-1}]$$

$$\begin{aligned} x &\longleftrightarrow y \\ x^{-1} &\longleftrightarrow y^{-1} \end{aligned}$$

Define $Z := X \underset{\varphi}{\cup} Y$ glued along φ. (see homework on glueing schemes)



These are non separated points (do not satisfy the valuative criterion).

For simplicity, assume k is algebraically closed

In $Z \times_k Z$, we have 4 nonseparated points:

(recall: in this case, the set of closed points of $Z \times_k Z$ the product of the set of closed points of Z with itself.)

(O_1, O_1), (O_1, O_2), (O_2, O_1), (O_2, O_2)

The image of $\Delta : Z \longrightarrow Z \times_k Z$ contains (O_1, O_1), (O_2, O_2) but not (O_1, O_2) and (O_2, O_1) which are in its closure.

Details: exercise: hint: construct the fiber product:

$$Z \times_k Z = X \times_k X \cup X \times_k Y \cup Y \times_k X \cup Y \times_k Y$$

<u>Lemma:</u> A morphism $f: X \to Y$ is separated if and only if the image of Δ is a closed subset of $X \times_Y X$.

<u>Proof:</u> If Δ is a closed embedding, its image is a closed subset of $X \times_Y X$.

Conversely, assume the image of Δ is closed.

$$\begin{tikzcd}
X \arrow[dr, "\Delta"] \arrow[drr, bend left, "Id_X"] \arrow[ddr, bend right, "Id_X"'] & & \\
& X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

$\Delta(X) \subset X \times_Y X$ is closed.

<u>Claim</u>: $\Delta : X \longrightarrow \Delta(X)$ is a homeomorphism

because $p_1 \circ \Delta = Id_X$.

Next we need to see that $\Delta^\# : \mathcal{O}_{X \times_Y X} \longrightarrow \Delta_* \mathcal{O}_X$

is surjective. From past homework, we know that

$\Delta^\#$ is surjective iff it is surjective on the stalks.

Choose $x \in X$ $\quad \Delta^\#(x) : \mathcal{O}_{X \times X, \Delta(x)} \longrightarrow \mathcal{O}_{X, x}$

Choose an open affine neighborhood $U = \operatorname{Spec} A$ of x.

Suppose U small enough so that $f(U) \subset \underset{\text{open}}{\operatorname{Spec} B} \subset Y$

$U \times_V U \subset X \times_Y X$ is an open affine neighborhood of $\Delta(x)$.

We saw that $\Delta|_U : U \longrightarrow U \times_Y U$ is a closed embedding (every morphism of affine schemes is separated)

$$\mathcal{O}_{U \times_Y U, \Delta(x)} = \mathcal{O}_{X \times_Y X, \Delta(x)}$$

$$\mathcal{O}_{U,x} = \mathcal{O}_{X,x}$$

$\Rightarrow$

$$\begin{array}{ccc} \mathcal{O}_{U \times_Y U, \Delta(x)} & \longrightarrow & \mathcal{O}_{U,x} \\ \| & & \| \\ \mathcal{O}_{X \times_Y X, \Delta(x)} & \longrightarrow & \mathcal{O}_{X,x} \end{array}$$

$\mathcal{O}_{U \times_Y U, \Delta(x)} \to \mathcal{O}_{U,x}$ is surjective because $\Delta|_U$ is a closed embedding

$\square$