Valuative criterion: Uses valuation rings. (Atiyah–McDonald)

Def: K a field. A subring R of K is called a valuation ring if $\forall\ x \neq 0,\ x \in K$, either $x \in R$ or $x^{-1} \in R$.

Main results about valuation rings:

① Valuation rings are local rings.

② Given two local subrings $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ of K, we say $(A, \mathfrak{m})$ dominates $(B, \mathfrak{n})$ if $B \subset A$ and

$$\mathfrak{m} \cap B = \mathfrak{n}.$$

This defines a partial order among local subrings of K.

Valuation rings are exactly the maximal elements of the set of local subrings of K for the dominance relation.

(3) Let Γ be a totally ordered abelian group.
A valuation v of K with values in Γ is a map

$$v : K^* \longrightarrow \Gamma \quad \text{s.t.}$$

(a) $v(xy) = v(x) + v(y)$

(b) $v(x+y) \geq \min(v(x), v(y))$

The set of elements $x \in K^*$ s.t. $v(x) \geq 0 \in \Gamma$ $(\cup \{0\})$
is a valuation ring of K with maximal ideal

$$\{x \in K \mid x = 0 \text{ or } v(x) > 0\}$$

The valuation v is called discrete if $\Gamma = \mathbb{Z}$.

The valuative criterion for separatedness:

Notation: K a field, $R \subset K$ a valuation ring.

$\mathfrak{m} \subset R$ the maximal ideal of R

$T := \operatorname{Spec} R$ has exactly one closed point: $\mathfrak{m}$

$U := \operatorname{Spec} K$ one-pointed scheme

$R \hookrightarrow K \iff U = \operatorname{Spec} K \hookrightarrow \operatorname{Spec} R = T$

image is the generic point of T (see homework) $\leftrightarrow (0) \subset R$

$(0) \subset \mathfrak{p} \ \forall \ \mathfrak{p} \in \operatorname{Spec} R \Rightarrow \overline{\{(0)\}} = T$

$\mathcal{O}_{T,\mathfrak{m}} = R_{\mathfrak{m}} = R \qquad \mathcal{O}_{T,(0)} = R_{(0)} = K$

<u>Theorem</u> (valuative criterion of separatedness)

Let $f : X \longrightarrow Y$ be a morphism of schemes. Assume X is noetherian. Then f is separated if and only if the following condition holds.

For any K and R as above, and any morphisms $T \to Y$, $U \to X$ forming the commutative diagram

$$\begin{tikzcd}
(0) \arrow[d, mapsto] & \operatorname{Spec} K = U \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
(0) \in \operatorname{Spec} R = & T \arrow[r] \arrow[ur, dotted, "i"] & Y
\end{tikzcd}$$

there is at most one morphism $i : T \to X$ making the whole diagram commutative.

For the proof, we need: (similar to homework)

<u>Lemma:</u> To give a morphism from $U = \operatorname{Spec} K$ to a scheme X is equivalent to giving a point $x_1 \in X$ and an inclusion of fields $k(x_1) \hookrightarrow K$. To give a morphism

from $T = \operatorname{Spec} R$ to X is equivalent to giving two points $x_0, x_1 \in X$, with $x_0 \in Z := \overline{\{x_1\}}$ and an inclusion of fields $k(x_1) \hookrightarrow K$ s.t. R dominates the local ring $\mathcal{O}_{Z, x_0}$, where Z is endowed with its reduced induced scheme structure.

Proof: The part about $U \to X$ was done in homework.

For the second part, let $t_0 (= \mathfrak{m}_R)$ be the closed point of T and $t_1 (= (0))$ the generic point of T.

Given a morphism $\varphi : T \to X$, let x_0 and x_1 be the images of t_0 and t_1. $Z := \overline{\{x_1\}}$

Since T is reduced, $T \xrightarrow{\varphi} X$ factors through X_{red}.

$$\begin{tikzcd} & T \arrow[r, "\varphi"] \arrow[d] & X \\ Z \arrow[r, hook] & X_{red} \arrow[ur] & \end{tikzcd}$$

Claim: $T \xrightarrow{\varphi} X$ factors through Z.

as sets: $T \xrightarrow{\varphi} X$ factors through Z.

$$\begin{tikzcd} T \arrow[rr, "\varphi"] \arrow[dr, "\tilde{\varphi}"'] & & X \\ & Z \arrow[ur, hook, "j"'] & \end{tikzcd}$$

on the sheaves:

$$\begin{tikzcd} \mathcal{O}_X \arrow[rr, "\varphi^\#"] \arrow[dr, "j^\#"'] & & \varphi_* \mathcal{O}_T \\ & j_* \mathcal{O}_Z \arrow[ur, dotted, "?"'] & \end{tikzcd}$$

The factorization will exist when $\varphi^\#$ is zero on the kernel of $j^\#$: see exercise II.3.11

Assuming this, we have $T \to Z \hookrightarrow X$

$\Rightarrow$ local hom. of local rings $\mathcal{O}_{Z,x_0} \to \mathcal{O}_{T,t_0} = R_{\mathfrak{m}} = R$

$\Rightarrow$ $(R, \mathfrak{m})$ dominates $(\mathcal{O}_{Z,x_0}, \mathfrak{m}_{Z,x_0}) \subset K$

$\mathcal{O}_{Z,x_0} \hookrightarrow R \subset K$

$\mathfrak{m} \cap \mathcal{O}_{Z,x_0} \leftarrow \cdots \mathfrak{m}$

Remark: $\underbrace{\operatorname{Spec} R = T \hookrightarrow Z}_{\text{integral schemes: have generic points: } t_1, x_1} \hookrightarrow X$

$$\mathcal{O}_{Z,x_1} = \text{field} \supset \mathcal{O}_{Z,z} \quad \forall z \in Z$$

from the first part $k(x_1) \subset K$

verify $\quad k(x_1) = \mathcal{O}_{Z,x_1} \subset K$

verify $\quad \mathcal{O}_{Z,x_0} \subset R = \mathcal{O}_{T,t_0}$, with $\mathcal{O}_{Z,x_0} \subset \mathcal{O}_{Z,x_1}$ and $R \subset K$

R dominates $\mathcal{O}_{Z,x_0}$ means $m_R \cap \mathcal{O}_{Z,x_0} = m_{Z,x_0}$.

Proof of lemma continued:

Conversely, suppose we are given $x_0, x_1 \in X$, $x_0 \in Z = \overline{\{x_1\}}$ and the inclusion $k(x_1) \subset K$ s.t. R dominates $\mathcal{O}_{Z,x_0}$

The inclusion $\mathcal{O}_{Z,x_0} \hookrightarrow R$ gives a morphism

$$T \longrightarrow \operatorname{Spec} \mathcal{O}_{Z,x_0}$$ compose this with

$$\operatorname{Spec} \mathcal{O}_{Z,x_0} \longrightarrow X :$$

Given any affine neighborhood $U = \operatorname{Spec} A \subset X$ of x_0, then U also contains x_1 because $x_0 \in \overline{\{x_1\}}$

$\iff$ $x_1 \in$ any open set containing x_0

$\left(x_1 \in X \setminus U \Rightarrow \overline{\{x_1\}} \subset X \setminus U \Rightarrow x_0 \in X \setminus U \right)$

$x_1 \in U \Leftarrow x_0 \in U$

We now have the evaluation map $A = \mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,x_0}$

$$\mathcal{O}_{X,x_0} \twoheadrightarrow \mathcal{O}_{Z,x_0}$$

$$\Rightarrow \begin{tikzcd}
\operatorname{Spec} \mathcal{O}_{Z,x_0} \arrow[r, hook] \arrow[rrd] & \operatorname{Spec} \mathcal{O}_{X,x_0} \arrow[r] & \operatorname{Spec} A = U \arrow[d, hook] \\
& & X
\end{tikzcd}$$

$$\Rightarrow T \to \operatorname{Spec} \mathcal{O}_{Z,x_0} \longrightarrow \operatorname{Spec} A \longrightarrow X \qquad \square$$