# Sheaves of modules:

Throughout, all schemes are noetherian.

(1) recall the definition of a sheaf of modules on a scheme $X$: it is a sheaf of abelian groups s.t. $\forall$ open $U \subset X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module.

and the module structures are compatible with the restriction morphisms: $\forall$ $V \subset U \subset X$ open sets

$$\forall \quad a \in \mathcal{O}_X(U), \ s \in \mathcal{F}(U) : \quad (as)\big|_V = a\big|_V \cdot s\big|_V$$

(2) A morphism of $\mathcal{O}_X$-modules $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves s.t. $\forall$ $U \subset X$ open, $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is a hom. of $\mathcal{O}_X(U)$-modules.

(3) The kernel, cokernel and image of a morphism of $\mathcal{O}_X$-modules is again an $\mathcal{O}_X$-module. The quotient of two $\mathcal{O}_X$-modules is an $\mathcal{O}_X$-module (cokernel of an inclusion).

(4) The direct sum, direct product, direct limit, inverse limit of $\mathcal{O}_X$-modules are again $\mathcal{O}_X$-modules.
(in these cases the presheaves are already sheaves, for direct limits, we need the noetherian hyp. Ex. in Section 1)

(5) <u>Tensor products</u>: Given $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, the tensor product $\mathcal{F} \otimes \mathcal{G}$ is the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. It is an $\mathcal{O}_X$-module.

(6) The sheaf $\mathcal{H}om$:

Given two $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, we have the set

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) := \left\{ \begin{matrix} \text{morphisms of } \mathcal{O}_X\text{-modules} \\ \mathcal{F} \longrightarrow \mathcal{G} \end{matrix} \right\}$$

The sheaf $\mathcal{H}om(\mathcal{F},\mathcal{G})$

the presheaf $\quad U \longmapsto \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$.

is an $\mathcal{O}_X$-module (and already a sheaf).

(7) An $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if there exist an open cover of $X$ by affine subschemes $U = \mathrm{Spec}\, A$ s.t. $\exists$ an $A$-module $M$ with $\mathcal{F}|_U \cong \tilde{M}$.

<u>recall</u>: $\tilde{M}$ is the sheaf on $\mathrm{Spec}\, A$ s.t. $\forall$ basic open $\mathrm{Spec}\, A[f^{-1}]$

$$\tilde{M}(\operatorname{Spec} A[f^{-1}]) = M[f^{-1}].$$

One can prove that then this holds for any open affine subset of $X$.

A quasi-coherent sheaf is called coherent if, in addition, $M$ is a finite $A$-module.

(8) An $\mathcal{O}_X$-module is called free if it is isomorphic, as an $\mathcal{O}_X$-module, to a direct sum of sheaves isomorphic to $\mathcal{O}_X$ (sometimes called trivial).

It is locally free if $X$ has a covering by open sets $U$ s.t. $\mathcal{F}|_U$ is free.

An isomorphism $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus I}$ is called a trivialization of $\mathcal{F}$ on $U$.

The rank of a locally free sheaf is the number of copies of $\mathcal{O}_U$ in a trivialization, it can be finite or infinite. The rank of a locally free sheaf is constant on each connected component of $X$.

Locally free sheaves are coherent when they have finite rank.

Glueing data: Suppose $\mathcal{F}$ is locally free and $X$ is connected. Let $X = \bigcup_{i=1}^{n} U_i$ be a covering of $X$ s.t. $\forall i$ (recall $X$ is noetherian so we can extract a finite subcover from any cover) $\exists \; \varphi_i : \mathcal{F}|_{U_i} \xrightarrow{\cong} \mathcal{O}_{U_i}^{\oplus I}$

On $U_{ij} := U_i \cap U_j$, we have the transition isomorphism $\varphi_{ij} := \varphi_j \varphi_i^{-1} : \mathcal{O}_{U_{ij}}^{\oplus I} \to \mathcal{F}|_{U_{ij}} \to \mathcal{O}_{U_{ij}}^{\oplus I}$

<u>Def</u>: An invertible sheaf is a locally free sheaf of rank 1.

If we put $\mathcal{L}^{-1} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$, then there is an isomorphism $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{-1} \cong \mathcal{O}_X$

More generally, for any $\mathcal{O}_X$-module $\mathcal{M}$, we have the natural morphism

$$\mathcal{M} \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X) \xrightarrow{\text{can.}} \mathcal{O}_X$$
$$s \otimes f \longmapsto f(s)$$

(define it as a morphism of presheaves, then factor through the sheafification)

When $\mathcal{M}$ is invertible, one shows that "can" is an isomorphism.

<u>Notation</u>: In general, we write $\mathcal{M}^* := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X)$

One can show that the tensor product defines a structure of abelian group on the set of isomorphism classes of invertible sheaves.

<u>Def</u>: This abelian group is called the Picard group, denoted $\mathrm{Pic}(X)$.

In the case of an invertible sheaf $\mathcal{L}$, the transition isomorphisms

$$\varphi_j \varphi_i^{-1} : \mathcal{O}_{U_{ij}} \xrightarrow{\cong} \mathcal{O}_{U_{ij}}.$$

are invertible elements of $\mathrm{Hom}(\mathcal{O}_{U_{ij}}, \mathcal{O}_{U_{ij}})$, hence are given by multiplication by invertible elements of $\mathcal{O}_X(U_{ij})$.

<u>Def</u>: These are the transition functions of $\mathcal{L}$.

<u>Exercise</u>: (1) Check that the transition functions of $\mathcal{L}^{-1}$ are the inverses of those of $\mathcal{L}$.

(2) Check that for two invertible sheaves $\mathcal{L}, \mathcal{L}'$, the transition functions of $\mathcal{L} \otimes \mathcal{L}'$ are the products of those of $\mathcal{L}$ and $\mathcal{L}'$ on a suitable cover.