

More generally: $R = k$ alg closed field

$$n=1, \text{ any } d, \quad \mathbb{P}^1 \hookrightarrow \mathbb{P}^m = \mathbb{P}^d \quad m = \binom{1+d}{d} - 1 = d$$

Def: The image is "the" rational normal curve of degree d . It is always the zero locus of some quadratic polynomials.

Example 2: Segre embeddings:

$$\mathbb{P}^l \times \mathbb{P}^m \xrightarrow{R} \mathbb{P}^n$$

where l, m are arbitrary, $n = (l+1)(m+1) - 1$.

In homogeneous coordinates (when $R = k$)
 $((a_0, \dots, a_l), (b_0, \dots, b_m)) \mapsto (a_0 b_0, a_0 b_1, \dots, a_l b_m)$
 $= (c_0, \dots, c_n)$

To define the morphism on the Proj:

$$\mathbb{T} := R[z_0, \dots, z_n]$$

$$S := R[X_0, \dots, X_l] \times_R R[Y_0, \dots, Y_m]$$

where, for two graded R -algebras R_1 and R_2 , the Cartesian R -product $R_1 \times_R R_2$ is, by definition,

$$R_1 \times_R R_2 := \bigoplus_{d \geq 0} R_{1,d} \otimes_R R_{2,d}$$

(Ex. II.5.11): There is a natural isomorphism

$$\text{Proj } R_1 \times_{\text{Spec } R} \text{Proj } R_2 \cong \text{Proj} (R_1 \times_R R_2)$$

$$\text{s.t. } f_1^* \mathcal{O}_{\text{Proj } R_1}(1) \otimes f_2^* \mathcal{O}_{\text{Proj } R_2}(1) \cong \mathcal{O}_{\text{Proj} (R_1 \times_R R_2)}(1)$$

Apply this to $R_1 = R[X_0, \dots, X_l]$, $R_2 = R[Y_0, \dots, Y_m]$
 then $S = R_1 \times_R R_2$

The elements $X_i \otimes Y_j \in R_1 \times_R R_2 = S$ form a set of generators for S as an R -algebra.

Choosing an ordering of $\{X_i \otimes Y_j\}$ means giving a bijection

$$\{X_i \otimes Y_j\} \longleftrightarrow \{Z_0, \dots, Z_n\}$$

which defines a surjective homomorphism of R -algebras

$$T = R[Z_0, \dots, Z_n] \twoheadrightarrow S = R_1 \times_R R_2$$

$$Z_p \longmapsto X_i \otimes Y_j$$

This defines a closed embedding $\mathbb{P}^l \times_{\text{Spec } R} \mathbb{P}^m \hookrightarrow \mathbb{P}^n$.

First case: The smooth quadric in \mathbb{P}^3 ($R = k$ alg closed field)

$$l = m = 1 \quad n = (l+1)(m+1) - 1 = 3$$

$$f: \mathbb{P}^1 \times_{\text{Spec } k} \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$$

$$(a_0, a_1), (b_0, b_1) \longmapsto (a_0 b_0, a_0 b_1, a_1 b_0, a_1 b_1) = (c_0, c_1, c_2, c_3)$$

relation: $c_0 c_3 - c_1 c_2$

$\Rightarrow \mathcal{L}(\mathbb{P}^1 \times \mathbb{P}^1) \subset \mathcal{Z}(z_0 z_3 - z_1 z_2)$ quadric
can show =

Understanding the \sim better in the projective case:

Recall that in the affine case: $\Gamma(\text{Spec} A, \tilde{M}) = M$
and we can recover \tilde{M} from M .

In the projective case: $T(\mathbb{P}_R^n, \mathcal{O}(d)) = S_d = S[d]_0$

where $S = R[X_0, \dots, X_n]$ $\underbrace{S[d]}_{\parallel}$

so we only recover the degree 0 piece and we cannot
recover the sheaf $\mathcal{O}(d) = \widetilde{S[d]}$.

Given a quasi-coherent sheaf \mathcal{F} , it would be nice if we
could find M s.t. $\tilde{M} \cong \mathcal{F}$.

Definition: For a sheaf \mathcal{F} on $\text{Proj } S$ (S any graded ring),
 put $\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}_{\text{Proj } S}(n)$ the "twist" of \mathcal{F} by n .

Define $\Gamma_*(\mathcal{F}) := \bigoplus_{n \in \mathbb{Z}} \Gamma(\text{Proj } S, \mathcal{F}(n))$

this is a graded module over S .

Remark: There is a natural morphism

$$\varphi: \widetilde{\Gamma_*(\mathcal{F})} \longrightarrow \mathcal{F}$$

defined on basic open sets as follows: $f \in S_m, m > 0$

$$\Gamma\left(\bigcup_f \subset \text{Proj } S, \widetilde{\Gamma_*(\mathcal{F})}\right) = \Gamma_*(\mathcal{F})[f^{-1}]_0 \xrightarrow{\quad ? \quad} \mathcal{F}\left(\bigcup_f\right)$$

given $s \in \Gamma_*(\mathcal{F})[f^{-1}]_0$, $\exists t \in \Gamma_*(\mathcal{F})$ and $d > 0$

s.t. $s = \frac{t}{f^d}$ and $\deg t = dm$, i.e., $t \in \Gamma(\mathcal{F}(dm))$

we define the image of s in $\Gamma(U_f, \mathcal{F})$ is, by def.,
the image of $t \otimes f^{-d}$ via the tensor product map
sections of presheaf to sections of associated sheaf

$$\Gamma(U_f, \mathcal{F}(d_m)) \otimes \Gamma(U_f, \mathcal{O}(-d_m)) \rightarrow \Gamma(U_f, \mathcal{F}(d_m)(-d_m))$$

$$t|_{U_f} \otimes (f|_{U_f})^{-d} \mapsto \varphi(s) \in \Gamma(U_f, \mathcal{F})$$

Prop. II.5.15: When S is finitely generated by S_1 , as
an S_0 -algebra and \mathcal{F} is quasi-coherent, the
morphism φ is an isomorphism.

Some important definitions:

Def 1: For any scheme X , $\mathbb{P}_X^n := \mathbb{P}_{\mathbb{Z}}^n \times_{\text{Spec } \mathbb{Z}} X$

($X \rightarrow \text{Spec } \mathbb{Z}$, obtained from $\mathbb{Z} \rightarrow \Gamma(X, \mathcal{O}_X)$)

The twisting sheaf $\mathcal{O}_{\mathbb{P}_X^n}(1)$ is, by def.,

$$\mathcal{O}_{\mathbb{P}_X^n}(1) := \mathcal{I}_1^* \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^n}(1)$$

Def 2: A morphism of schemes $W \xrightarrow{\downarrow} X$ is an embedding if it is a composition $W \hookrightarrow U \hookrightarrow X$ (or immersion) where $W \hookrightarrow U$ is a closed embedding and $U \hookrightarrow X$ is an open embedding.

Note: we cannot reverse the order open/closed: see example 29.3.4 in the Stacks project.
We can if W is reduced or g.c.

Def. 3: For any morphism of schemes $\pi: X \rightarrow Y$,
 an invertible sheaf \mathcal{L} on X is very ample relative to π
 if \exists an (π -)embedding $i: X \hookrightarrow \mathbb{P}_Y^n$ s.t.

$$\mathcal{L} \cong i^* \mathcal{O}_{\mathbb{P}_Y^n}(1)$$

Def. 4: Given a sheaf \mathcal{F} and a collection $\{s_i, i \in I\}$ of
 global sections of \mathcal{F} on X , we say \mathcal{F} is generated by
 $\{s_i, i \in I\}$ if, $\forall x \in X$, the stalk \mathcal{F}_x is generated
 as an $\mathcal{O}_{X,x}$ -module by the germs $\{s_i(x), i \in I\}$.

We say \mathcal{F} is generated by global sections, or, globally
 generated, if it is generated by some collection of
 global sections.

Remark: For any \mathcal{O}_X -module \mathcal{F} and collection

$\{s_i \mid i \in I\} \subset \Gamma(X, \mathcal{F})$, we can define a morphism of

\mathcal{O}_X -modules: $\varphi: \mathcal{O}_X^{\oplus I} \longrightarrow \mathcal{F}$

which on any open $U \subset X$ sends $\sum_{i \in I} f_i \in (\mathcal{O}_X^{\oplus I})(U)$

to $\sum_{i \in I} f_i s_i|_U \in \mathcal{F}(U)$.

We have that \mathcal{F} is generated by $\{s_i \mid i \in I\}$ iff

φ is surjective.

Another notion we need: Zeros of sections of (quasi-)coherent sheaves.

Let \mathcal{F} be a (quasi-)coherent sheaf on a noetherian scheme X and $s \in \Gamma(X, \mathcal{F})$. We define the scheme of zeros of s ,