Main Lemma: There is a natural isomorphism

$$g: Y \xrightarrow{\cong} \operatorname{Proj} S(Y)$$

Proof: The quotient morphism $\varphi: S \twoheadrightarrow S(Y) = S/I(Y)$ induces a morphism $f: \operatorname{Proj} S(Y) \to \operatorname{Proj} S$.

(f is defined everywhere because φ is surjective)

We will show that f factors through an isom.

$$\operatorname{Proj} S(Y) \xrightarrow{\cong} Y \hookrightarrow X := \operatorname{Proj} S$$

We have the exact sequence

$$0 \to I_Y \to S \to S(Y) \to 0 \quad \text{(by def.)}.$$

Lemma 1: The $\sim$ functor is exact.

Proof: Suppose given an exact sequence of graded S-modules: $0 \to M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$

$\forall d$, $\forall\ t \in S_d$, we have the exact sequences

$$0 \longrightarrow M'[t^{-1}] \longrightarrow M[t^{-1}] \longrightarrow M''[t^{-1}] \longrightarrow 0$$

and $0 \to M'[t^{-1}]_0 \longrightarrow M[t^{-1}]_0 \longrightarrow M''[t^{-1}]_0 \longrightarrow 0$

on the affine scheme $U_{(t)} := \operatorname{Spec} S[t^{-1}]_0 \subset X$

$$\Rightarrow \quad 0 \longrightarrow \widetilde{M'[t^{-1}]_0} \longrightarrow \widetilde{M[t^{-1}]_0} \longrightarrow \widetilde{M''[t^{-1}]_0} \longrightarrow 0$$

on $U_{(t)}$

$$\|$$

$$0 \longrightarrow \tilde{M}'\big|_{U_{(t)}} \longrightarrow \tilde{M}\big|_{U_{(t)}} \longrightarrow \tilde{M}''\big|_{U_{(t)}} \longrightarrow 0$$

The open sets $U_{(t)}$ cover $X \Rightarrow 0 \to \tilde{M}' \to \tilde{M} \to \tilde{M}'' \to 0$ is exact on X. $\square$

Proof of the main lemma:

We have the exact sequence

$$0 \longrightarrow I_Y \longrightarrow S \longrightarrow S(Y) \longrightarrow 0$$

$$\Rightarrow \quad (A) \qquad 0 \longrightarrow \widetilde{I}_Y \longrightarrow \widetilde{S} \longrightarrow \widetilde{S(Y)} \longrightarrow 0$$

and the exact sequence

$$(B) \qquad 0 \longrightarrow \mathcal{J}_Y \longrightarrow \mathcal{O}_X \longrightarrow i_* \mathcal{O}_Y \longrightarrow 0$$

where $i : Y \hookrightarrow X = \mathbb{P}^n_R = \operatorname{Proj} S$.

First, by the fact that $\widetilde{\Gamma_*(\mathcal{F})} \xrightarrow{\simeq} \mathcal{F}$ for coherent sheaves $\mathcal{F}$ on X, we have $\widetilde{I}_Y \cong \mathcal{J}_Y$ and $\widetilde{S} \cong \mathcal{O}_X$ and $I_Y \hookrightarrow S$ is the global sections of $\mathcal{J}_Y \hookrightarrow \mathcal{O}_X$

$\Rightarrow$ we have the commutative

$$\begin{tikzcd} \widetilde{I}_Y \arrow[r, hook] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & \widetilde{S} \arrow[d, equal] \\ \mathcal{J}_Y \arrow[r, hook] & \mathcal{O}_X \end{tikzcd}$$

$\Rightarrow$ we can form the commutative diagram with exact rows

$$\begin{tikzcd}
(A) & 0 \arrow[r] & \widetilde{I_Y} \arrow[r] \arrow[d, "\cong"] & \widetilde{S} \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "\Rightarrow"] & \widetilde{S(Y)} \arrow[r] \arrow[d, "\cong"] & 0 \\
(B) & 0 \arrow[r] & \mathcal{I}_Y \arrow[r] & \mathcal{O}_X \arrow[r] & i_* \mathcal{O}_Y \arrow[r] & 0
\end{tikzcd}$$

First consequence: $\operatorname{Support}(\widetilde{S(Y)}) = \operatorname{Support}(i_* \mathcal{O}_Y) = Y$

<u>Claim</u>: $\operatorname{support}(\widetilde{S(Y)}) = \{ \mathfrak{p} \in X \mid \mathfrak{p} \supset I_Y \}$

<u>Proof</u>:
$$\begin{aligned}
\operatorname{support}(\widetilde{S(Y)}) &= \{ \mathfrak{p} \in X \mid \widetilde{S(Y)}_{\mathfrak{p}} \neq 0 \} \\
&= \{ \mathfrak{p} \in X \mid (S(Y)_{\mathfrak{p}})_0 \neq 0 \} \\
&= \{ \mathfrak{p} \in X \mid S(Y)_{(\mathfrak{p})} \neq 0 \} \\
&= \{ \mathfrak{p} \in X \mid (S/I_Y)_{\mathfrak{p}} \neq 0 \}
\end{aligned}$$

Now: $\left(S/I_Y\right)_p = 0 \iff S_p = (I_Y)_p$

$\iff I_Y \setminus p \neq \emptyset$

$\iff I_Y \not\subset p$

hence $\quad \mathrm{Support}(\widetilde{S(Y)}) = \{ p \in X \mid I_Y \subset p \}$

$$\Rightarrow \quad Y = \mathrm{Support}(\widetilde{S(Y)}) = \{ p \in X \mid p \supset I_Y \}$$

$$= \mathrm{Proj}(S(Y)) \underset{\text{natural}}{\xhookrightarrow{j}} \mathrm{Proj}\, S$$

and
$$\begin{tikzcd}
i_* \mathcal{O}_Y \arrow[r, phantom, "\cong"] & \widetilde{S(Y)} \arrow[r, phantom, "\cong"] & j_* \mathcal{O}_{\mathrm{Proj}(S(Y))} \\
\mathcal{O}_X \arrow[u] \arrow[rr, equal] \arrow[urr, phantom, "\circlearrowleft"] & & \mathcal{O}_X \arrow[u]
\end{tikzcd}$$

$$\Rightarrow \quad Y = \mathrm{Proj}\, S(Y) \hookrightarrow \mathrm{Proj}\, S . \qquad \square$$

Divisors: The Zariski topology is coarse: algebraic subvarieties of an algebraic variety carry a lot of information about the ambient variety. So we want to study them. The simplest of these are those that have codimension 1: these give rise to the notion of Weil divisors. The ideal sheaves of codim. 1 subvarieties give rise to the notion of Cartier divisors.

These also help us understand morphisms from one scheme to another: we can pull back Cartier divisors, we can push forward Weil divisors.

Cartier divisors are closely related to invertible sheaves.

From now on, we will assume X is an integral and noetherian scheme, separated over $\operatorname{Spec}\mathbb{Z}$.

<u>Def</u>: An integral or prime Weil divisor is a closed subscheme Y of X which is integral of codimension 1.

<u>Note</u>: Y is the closure of its generic point, say $y \in X$.

<u>Lemma</u>: The local ring $\mathcal{O}_{Y,y}$ has dimension 1.

<u>Proof</u>: Choose an open affine $\operatorname{Spec} A \subset X$ s.t.

$$Y \cap \operatorname{Spec} A \neq \emptyset \Rightarrow y \in \operatorname{Spec} A$$

$Y \cap \operatorname{Spec} A \subset \operatorname{Spec} A$ is a closed subscheme, integral and of codim. 1.

$y \longleftrightarrow \mathfrak{p} \subset A$ prime of height 1.

$$0 \to \mathfrak{p} \to A \to A/\mathfrak{p} \to 0$$

and localize at $\mathfrak{p}$: $0 \to \mathfrak{p}A_{\mathfrak{p}} \to A_{\mathfrak{p}} \to (A/\mathfrak{p})_{\mathfrak{p}} \to 0$

local ring $\mathcal{O}_{X,y}$ $\qquad$ $\parallel$ residue field of y

any chain of prime ideals of $A_{\mathfrak{p}}$ is contained in $\mathfrak{p}A_{\mathfrak{p}}$ $\Rightarrow$ $\exists$ only one chain $0 \subset \mathfrak{p}A_{\mathfrak{p}}$ ($\mathfrak{p}$ has height 1)

$\Rightarrow$ $A_{\mathfrak{p}}$ has dim. 1. $\square$

So integral Weil divisors in X are in 1-to-1 correspondence with the points of X whose local rings have dim. 1.

<u>Def</u>: A Weil divisor on X is a formal linear combination of integral Weil divisors with integer coefficients. The set of all Weil divisors is the free abelian group with basis

all the integral Weil divisors:

$$\mathrm{Div}(X) := \left\{ \sum_{\text{finite}} n_Y [Y] \;\middle|\; \begin{array}{l} n_Y \in \mathbb{Z} \\ Y \subset X \text{ integral} \\ \text{Weil div.} \end{array} \right\}$$

Example: If X has dimension 1, then integral Weil divisors are the closed points of X. Weil divisors are of the form $\displaystyle\sum_{\substack{P \in X \\ \text{closed pt}}} n_P [P]$.

We need to impose more restrictions on X, in order to have a nice relation between Weil and Cartier divisors.

We need a fundamental result in commutative algebra: Nakayama's lemma.

Def: The Jacobson radical $J(A)$ of a ring A is the intersection of all the maximal ideals of A.

Proposition (Nakayama's lemma): $\mathfrak{a} \subset A$ an ideal, suppose $\mathfrak{a} \subset J(A)$. $\forall M$ finitely generated A-module,

if $\mathfrak{a} M = M$, then $M = 0$.

Consequence 1: If $N \subset M$ is a submodule,

$$M = \mathfrak{a} M + N \Rightarrow M = N \quad \text{(apply the prof. to } M/N\text{)}$$

Consequence 2: If A is a local ring with maximal ideal $\mathfrak{m}$, and $x_1, \dots, x_n$ are elements of M, then if the images of $x_1, \dots, x_n$ in $M/\mathfrak{m}M$ generate it, $x_1, \dots, x_n$ generate M.

We will have a nice relation between Weil and Cartier divisors when X is "regular in codim. 1".

This means all the 1-dim. local rings of X are "regular"

<u>Def</u>: A finite dimensional noetherian local ring with maximal ideal m is regular if $\dim R$ is equal to the minimal number of generators of m.

Equivalently, by Consequence 2, $\dim R = \dim_k m/m^2$

where $k = R/m$

$$\left(m/m^2 \cong m \otimes_R R/m \cong m \otimes_R k \right)$$

<u>Def</u>: A discrete valuation ring (DVR) is a noetherian regular local ring of dim. 1.

Facts: If R is a DVR, then $\mathfrak{m}$ is principal.

A generator π of $\mathfrak{m}$ is called a uniformizer.

Any ideal of R is a power of $\mathfrak{m}$ $\Rightarrow$ R is a PID.

For any $f \in R$, $\exists$ $n \geq 0$ and $u \in R^\times$ (unit)

s.t. $f = u\pi^n$

If $K := \mathrm{Frac}(R)$, then $\forall$ $g \in K$, $\exists$ $u \in R^\times$

and $n \in \mathbb{Z}$ s.t. $g = u\pi^n$.

The map $g \longmapsto \underbrace{v_R(g) := n}_{\in \mathbb{Z}}$ is a valuation of K

with valuation ring R: this is a discrete valuation.

Hypothesis (∗): X is noetherian, integral, separated, regular in codim. 1.

In particular, all the 1-dim. local rings of X are DVRs.

We define an equivalence relation on Weil divisors.

Let K be the function field of X: this is the local ring of the generic point of X and the field of fractions of the ring A of any affine open $\operatorname{Spec} A \subset X$.

We refer to the elements of K as rational functions on X.