By construction, $\mathcal{O}_X(D) \cong \mathcal{O}_X \iff D$ is principal

<u>Also note</u>: multiplication of Cartier divisors

$\longleftrightarrow$ tensor product of invertible sheaves

(<u>locally</u>: $\mathcal{O}_{U_i} f_i^{-1} \otimes_{\mathcal{O}_{U_i}} \mathcal{O}_{U_i} g_i^{-1} \cong \mathcal{O}_{U_i} f_i^{-1} g_i^{-1}$

$=$ as subsheaves of $\mathcal{K}_{U_i}$)

$-D$ represented by $\{(f_i^{-1}, U_i)\}$ if D is rep. by $\{(f_i, U_i)\}$

$\Rightarrow \mathcal{O}_X(-D)$ is locally $\mathcal{O}_{U_i} f_i$

$\Rightarrow \quad \mathcal{O}_X(D) \cong \mathcal{O}_X(D') \iff \mathcal{O}_X(D) \otimes \mathcal{O}_X(D')^{-1} \cong \mathcal{O}_X$

$\iff \mathcal{O}_X(D) \otimes \mathcal{O}_X(-D') \cong \mathcal{O}_X \iff \mathcal{O}_X(D-D') \cong \mathcal{O}_X$

$\iff D-D'$ is principal $\iff D$ is linearly eq. to D'.

We have $\mathrm{Pic}(X) :=$ group of invertible sheaves on X modulo isomorphism.

$\mathrm{Pic}^{\mathrm{rat}}(X) :=$ group of invertible subsheaves of $\mathcal{K}_X$ modulo isomorphism

$\mathrm{CaCl}(X) :=$ group of Cartier divisors on X modulo linear equivalence.

We just saw that $\mathrm{CaCl}(X) \cong \mathrm{Pic}^{\mathrm{rat}}(X)$
$(\hookrightarrow \mathrm{Pic}(X))$

<u>Lemma:</u> If X is integral, then
$$\mathrm{Pic}^{\mathrm{rat}}(X) = \mathrm{Pic}(X).$$

<u>Proof:</u> We need to show that every invertible sheaf is isomorphic to an invertible subsheaf of $\mathcal{K}_X$.

Let $\mathcal{L}$ be an invertible sheaf.

$$\mathcal{O}_X \hookrightarrow \mathcal{K}_X$$

$$\Rightarrow \quad \begin{array}{c} \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{L} \\ \wr\| \\ \mathcal{L} \end{array} \longrightarrow \mathcal{K}_X \otimes \mathcal{L}$$

Claim $\mathcal{K}_X \otimes_{\mathcal{O}_X} \mathcal{L} \cong \mathcal{K}_X$, because

X is integral $\Rightarrow$ $\mathcal{K}_X$ is the constant sheaf with group $K(X) = \mathcal{O}_{X,\eta}$ η = generic point

we verify that $\mathcal{K}_X \otimes_{\mathcal{O}_X} \mathcal{L}$ is also the constant sheaf with group $K(X)$.

On any open set U where $\mathcal{L}$ is trivial, we have $(\mathcal{K}_X \otimes \mathcal{L})|_U \cong \mathcal{K}_X|_U \otimes \mathcal{L}|_U \cong \mathcal{K}_X|_U \cong \mathcal{K}_U$

$\mathcal{K}_U$ is the constant sheaf on U with group $K(U)=K(X)$.

If $V \subset X$ is any open set, then $V = \bigcup_{i \in I} V \cap U_i$

where U_i is affine and $\mathcal{L}|_{U_i} \simeq \mathcal{O}_{U_i}$.

Then if $s \in \Gamma(V, \mathcal{K}_X \otimes \mathcal{L})$

$$s \Longleftrightarrow \{ s_i := s|_{V \cap U_i} \}$$

$$s_i|_{V \cap U_i \cap U_j} = s_j|_{V \cap U_i \cap U_j} = s|_{V \cap U_i \cap U_j} \in K(X) = K(V \cap U_i \cap U_j).$$

$$\Rightarrow s \in K(X)$$

$$\Rightarrow \mathcal{K}_X \otimes \mathcal{L}(V) = K(X). \qquad \square$$

Cartier versus Weil divisors.

Proposition: Suppose X is integral, separated, locally factorial (i.e., all local rings of X are UFD). Then, the additive group $\mathrm{Div}(X)$ is isomorphic to the multiplicative group $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$.

Under this isomorphism, principal divisors map to principal divisors, i.e., $Cl(X) \cong CaCl(X)$ $(\cong \mathrm{Pic}(X))$

Proof: We construct morphisms of groups

$$D : \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \longrightarrow \mathrm{Div}(X)$$

$$F : \qquad \mathrm{Div}(X) \longrightarrow \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$$

which are inverses of each other and send principal divisors to principal divisors.

__Definition of D:__ X is integral, so $\mathcal{K}_X$ is the constant sheaf with group $K(X)$. Let C be a Cartier divisor represented by $\{(f_i, U_i), X = \bigcup_{i\in I} U_i, f_i \in K(X)\}$

$$D(C) := \sum_{\substack{Y\subset X \\ \text{integral divisor}, \, i \text{ s.t. } Y\cap U_i \neq \emptyset}} v_Y(f_i)\,[Y]$$

this is well-defined because if $Y \cap U_i \cap U_j \neq \emptyset$, then

$$f_i/f_j \in \mathcal{O}^*(U_i \cap U_j) \Rightarrow v_Y(f_i/f_j) = 0$$
$$\Rightarrow v_Y(f_i) = v_Y(f_j).$$

the sum is finite because X is noetherian: we can cover X with a finite number of the U_i and on each U_i, only a finite # of $v_Y(f_i)$ are nonzero. __Note__: we only used X integral noetherian.

Definition of F: We first need the following lemma:

Lemma: Suppose X is integral and locally factorial. Then every integral Weil divisor Y on X is locally principal. In other words, every $x \in X$ has an affine neighborhood $U = \operatorname{Spec} A$ s.t. the ideal of $Y \cap U$ in U is principal.

Proof of the lemma: If $x \notin Y$, then we can find $U = \operatorname{Spec} A$ s.t. $U \cap Y = \phi$, then $I_{Y \cap U} = A$ is principal.

Suppose $x \in Y$ and let $a_1, \dots, a_n$ be a generating set for $I_{Y \cap U}$. Let $\mathfrak{q} \subset A$ be the prime ideal corresponding to x.

Then $\mathcal{O}_{X,x} = A_{\mathfrak{q}}$.

Since Y is an integral Weil divisor, $I_{Y\cap U}$ is prime of height 1. The germs $a_1(x), \dots, a_n(x)$ generate $\mathfrak{p} := I_{Y\cap U} A_{\mathfrak{q}}$ which is also prime of height 1.

Since $A_{\mathfrak{q}}$ is a UFD, the $a_i(x)$ are products of irreducible elements. $\forall i$ one of the irreducible factors of $a_i(x)$, say b_i, belongs to $\mathfrak{p}$. $\underset{\uparrow\,\text{prime}}{\langle b_i \rangle} \subset \mathfrak{p}$, $\mathfrak{p}$ has height 1

$$\Rightarrow \quad \langle b_i \rangle = \mathfrak{p} \quad \forall i \quad \text{and} \quad a_i(x) = \text{unit}\, b_i$$

So $\exists \; g_i, h_i, d \in A \setminus \mathfrak{q}, \; c \in A$ s.t. $a_i(x) = \dfrac{g_i}{h_i} \dfrac{c}{d} \quad \forall i$

Replacing A with $A[g_i^{-1}, h_i^{-1}, d^{-1}]_{\text{all } i}$, we obtain that $I_{Y\cap U} = \langle c \rangle$ is principal. $\square$

Back to the definition of F:

Let $W = \sum_{i=1}^{m} n_i Y_i$ be a Weil divisor. We will find a Cartier divisor $\{(f_U, U)\}$ whose associated Weil divisor is W.

Let $x \in X$ be a point. Choose an affine neighborhood $U = \operatorname{Spec} A$ of x s.t. if $x \notin Y_i$, then $U \cap Y_i = \phi$.

By the lemma, we can also choose U small enough so that the ideals $I_{Y_i \cap U}$ are principal $\forall i$, say $I_{Y_i \cap U} = \langle a_i \rangle = A a_i$

We have that $W \cap U = \sum_{i=1}^{m} n_i [Y_i \cap U]$

is the divisor of $f_U := \prod_{i=1}^{m} a_i^{n_i}$, i.e., $W \cap U = D(f_U)$.

Because: the image of a_i in the local ring of the generic point η_i of Y_i generates the maximal ideal of $\mathcal{O}_{X,\eta_i}$

because $(a_i) = I_{Y_i \cap U}$ $\quad (a_i) A_{(a_i)} = m_{X,\eta_i}$.

We have a well-defined Cartier divisor $F(W)$ represented by $\{(U, f_U)\}$, because, for two affine open sets U, V as above, f_U / f_V is invertible: $U \cap V$ is affine because X is separated, f_U and f_V generate the same ideal in the ring of $U \cap V$.

By construction the maps D and F are inverses to each other and principal divisors go to principal divisors (think about this a little bit at home). $\square$

<u>Remark</u>: The morphism D is well-defined without assuming X locally factorial.

The proof above shows that we can think of Cartier divisors as locally principal Weil divisors (i.e., every point has a neighborhood on which the Weil divisor is principal).

What we proved is that when X is locally factorial, all Weil divisors are locally principal.

This is "the same" as saying that $Cl(\operatorname{Spec} A) = 0$ when A is a UFD.