__Example:__ $\mathbb{P}^n_k$ is locally factorial.

$$\Rightarrow \quad
\begin{tikzcd}
\mathrm{Cl}(\mathbb{P}^n_k) \arrow[r, phantom, "\cong"] \arrow[d, equal] & \mathrm{CaCl}(\mathbb{P}^n_k) \arrow[r, phantom, "\cong"] & \mathrm{Pic}(\mathbb{P}^n_k) \arrow[d, equal] \\
\mathbb{Z}[Z_0] \arrow[r, phantom, "="] & \mathbb{Z}[Z(X_0)] \arrow[r, phantom, "="] & \mathbb{Z}[\mathcal{O}_{\mathbb{P}^n}(1)] \arrow[ull, dashed]
\end{tikzcd}$$

__Effective divisors:__

__Def:__ A Cartier divisor is effective if it can be represented by $\{(f_i, U_i)\}$ s.t. $\forall i$ $f_i \in \mathcal{O}(U_i) \subset \mathcal{K}_X(U_i)$.

A Weil divisor $C = \sum_{i=1}^{m} n_i [Y_i]$ is effective if $\forall i, n_i \geq 0$.

The proof of the previous proposition shows that on a noetherian, integral, separated, locally factorial scheme, effective Cartier divisors correspond to effective Weil divisors.

Given an effective Cartier divisor C def. by $\{(U_i, f_i)\}$, we define the associated locally principal subscheme $Z(C)$ to be the subscheme whose sheaf of ideals is generated by f_i on U_i: $\mathcal{I}_{Z(C)\cap U_i} := \mathcal{O}_{U_i} f_i \subset \mathcal{O}_{U_i}$.

In the proof of the proposition, we had $f_U = \prod_{i=1}^{m} a_i^{n_i}$ and

$$I_{Z(C)\cap U} = \langle f_U \rangle \subset A.$$

These glue together to define $\mathcal{I}_{Z(C)} \subset \mathcal{O}_X$ because on any $U_i \cap U_j$: f_i / f_j is invertible $\Rightarrow$

$$\mathcal{O}_{U_i\cap U_j} f_i = \mathcal{O}_{U_i \cap U_j} f_j \subset \mathcal{O}_{U_i\cap U_j}.$$

Recall that we defined $\mathcal{O}_X(C)$ as the sub-sheaf of $\mathcal{K}_X$ generated on U_i by f_i^{-1}.

This means $\mathcal{I}_{Z(C)} = \mathcal{O}_X(-C)$ by def.

when C is effective. So $\mathcal{O}_X(-C) \subset \mathcal{O}_X \subset \mathcal{O}_X(C) \subset \mathcal{K}_X$

(locally $\mathcal{O}_{U_i} \cdot f_i \subset \mathcal{O}_{U_i} \subset \mathcal{O}_{U_i} \cdot f_i^{-1} \subset \mathcal{K}_{U_i}$)

Morphisms to projective space: (everything noetherian)

We fix a ring A, $\mathbb{P}^n := \mathbb{P}^n_A$

Theorem: Suppose X is a scheme over A ($X \to \operatorname{Spec} A$)

(1) Given an invertible sheaf $\mathcal{L}$ on X and global sections $s_0, \dots, s_n$ of $\mathcal{L}$ which generate $\mathcal{L}$, there exists a unique A-morphism $\varphi : X \to \mathbb{P}^n_A$ s.t. $\mathcal{L} \cong \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr] & \arrow[d, phantom, "\circlearrowleft"] & \mathbb{P}^n_A \arrow[dl] \\ & \operatorname{Spec} A & \end{tikzcd}$$

and $\forall i$ $\quad s_i = \varphi^* x_i$.

(2) Given an A-morphism $\varphi : X \longrightarrow \mathbb{P}^n$, put $\mathcal{L} := \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$ and $s_i := \varphi^* X_i$ $\forall i = 0, \dots, n$. Then the sections s_i generate $\mathcal{L}$ and φ is the morphism from (1) associated to $\mathcal{L}$ and $s_0, \dots, s_n$.

Proof: (1) For each i, let

$$V_i := \{ x \in X \mid s_i(x) \notin m_x \mathcal{L}_x \}$$

be the open set of X where s_i generates $\mathcal{L}$ (i.e., $s_i(x)$ generates $\mathcal{L}_x$): $\forall x \in V_i$ $\mathcal{L}_x = \mathcal{O}_{X,x} s_i(x)$. ($m_x \subset \mathcal{O}_{X,x}$)

$$\Rightarrow \mathcal{L}|_{V_i} \xleftarrow{\cong} \mathcal{O}_{V_i} \, s_i|_{V_i}$$

Fix i, $\forall j$ $\exists\, t_{ji} \in \mathcal{O}_X(V_i)$ s.t. $s_j|_{V_i} = t_{ji} \, s_i|_{V_i}$

Define $\varphi_i : V_i \longrightarrow U_i = \operatorname{Spec} A\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right] \subset \mathbb{P}^n$

by the morphism of global sections (of A-alg.)

$$\begin{aligned} \varphi_i^{\#} : A\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right] &\longrightarrow \mathcal{O}_X(V_i) \\ & \qquad\quad \cup\!\!\!\!\!\;\;| \\ \frac{X_j}{X_i} &\longmapsto t_{ji} \end{aligned}$$

Claim: The morphisms φ_i glue together to give a morphism $\varphi : X \longrightarrow \mathbb{P}^n$.

We need to verify that $\varphi_i \big|_{V_i \cap V_j} = \varphi_j \big|_{V_i \cap V_j}$

$$V_i \cap V_j \underset{\varphi_j}{\overset{\varphi_i}{\rightrightarrows}} U_i \cap U_j = \operatorname{Spec} A\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right]\left[\left(\frac{X_j}{X_i}\right)^{-1}\right]$$

$$= \operatorname{Spec} A\left[\frac{X_0}{X_j}, \dots, \frac{X_n}{X_j}\right]\left[\left(\frac{X_i}{X_j}\right)^{-1}\right]$$

The identification between the two rings:

$$A\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right]\left[\left(\frac{X_j}{X_i}\right)^{-1}\right] \xleftrightarrow{\cong} A\left[\frac{X_0}{X_j}, \dots, \frac{X_n}{X_j}\right]\left[\left(\frac{X_i}{X_j}\right)^{-1}\right]$$

$$\forall l \qquad \frac{X_l}{X_i} \longleftrightarrow \frac{X_l}{X_j}\left(\frac{X_i}{X_j}\right)^{-1}$$

$$\frac{X_l}{X_i} \cdot \left(\frac{X_j}{X_i}\right)^{-1} \longleftrightarrow \frac{X_l}{X_j}$$

$$\varphi_i^{\#}\big|_{V_i \cap V_j} : \quad \frac{X_l}{X_i} \longmapsto t_{li} \qquad \left(\frac{X_j}{X_i}\right)^{-1} \longmapsto (t_{ji})^{-1}$$

$$\varphi_j^{\#}\big|_{V_i \cap V_j} : \quad \frac{X_l}{X_j} \longmapsto t_{lj} \qquad \left(\frac{X_i}{X_j}\right)^{-1} \longmapsto (t_{ij})^{-1}$$

We need to verify that $\varphi_i^{\#}\left(\frac{X_l}{X_i}\right) = \varphi_j^{\#}\left(\frac{X_l}{X_j}\left(\frac{X_i}{X_j}\right)^{-1}\right)$

(and similarly after switching i and j)

$$\varphi_i^{\#}\left(\frac{X_l}{X_i}\right) = t_{li} \qquad \varphi_j^{\#}\left(\frac{X_l}{X_j}\left(\frac{X_i}{X_j}\right)^{-1}\right) = t_{lj} \cdot (t_{ij})^{-1}$$

<u>Claim</u>: $(t_{ji})^{-1} = t_{ij}$ and $\forall\, i, j, l \quad t_{ij} \cdot t_{jl} = t_{il}$

Note that the claim will imply $\varphi_i^{\#}\big|_{V_i \cap V_j} = \varphi_j^{\#}\big|_{V_i \cap V_j}$

Proof of the claim: On $V_i \cap V_j$:

$$\mathcal{L}|_{V_i\cap V_j} = \mathcal{O}_{V_i\cap V_j} s_i|_{V_i\cap V_j} = \mathcal{O}_{V_i\cap V_j} s_j|_{V_i\cap V_j}$$

and $s_i|_{V_i\cap V_j} = t_{ij}|_{V_i\cap V_j} s_j|_{V_i\cap V_j} = t_{ij}|_{V_i\cap V_j} t_{ji}|_{V_i\cap V_j} s_i|_{V_i\cap V_j}$

s_i is a generator of a free rank 1 module

$$\Rightarrow t_{ij}|_{V_i\cap V_j} t_{ji}|_{V_i\cap V_j} = 1$$

$$s_l|_{V_i\cap V_j} = t_{li}|_{V_i\cap V_j} s_i|_{V_i\cap V_j} = t_{lj}|_{V_i\cap V_j} s_j|_{V_i\cap V_j}$$

$$= t_{lj}|_{V_i\cap V_j} t_{ji}|_{V_i\cap V_j} s_i|_{V_i\cap V_j}$$

$$\Rightarrow \quad t_{li}\big|_{V_i \cap V_j} = t_{lj}\big|_{V_i\cap V_j}\, t_{ji}\big|_{V_i \cap V_j} \quad \checkmark$$

So the φ_i glue to give $\varphi : X \longrightarrow \mathbb{P}^n$

Note that on U_i : $\quad \mathcal{O}_{\mathbb{P}^n}(1)\big|_{U_i} = \mathcal{O}_{U_i}\, X_i$

when we pull back $\varphi_i^*\left(\mathcal{O}_{\mathbb{P}^n}(1)\big|_{U_i}\right) = \varphi_i^*\, \mathcal{O}_{U_i} X_i$

$$= \mathcal{O}_{V_i}\, \varphi_i^* X_i$$

So, on V_i $\quad \varphi^*\mathcal{O}_{\mathbb{P}^n}(1)\big|_{V_i} = \varphi_i^*\left(\mathcal{O}_{\mathbb{P}^n}(1)\big|_{U_i}\right)$

is trivial.

The transition functions of $\varphi^*\mathcal{O}_{\mathbb{P}^n}(1)$ are $\varphi^*\left(\frac{X_i}{X_j}\right)$

we defined $\varphi^*\left(\frac{X_i}{X_j}\right) = \varphi^\#\left(\frac{X_i}{X_j}\right) = \varphi^\#\left(\frac{X_i}{X_j}\right) = t_{ij}$