Differentials :

We want to define vector fields algebraically.

In differential geometry, vector fields or tangent vectors act on functions.

Given a function f in a neighborhood of a point x and a tangent vector v at x, $v(f) = v(f - \text{constant})$

$$= v(f - f(x)).$$

So we concentrate on functions that vanish at x.

Algebraically, given a scheme X and a point $x \in X$, we think of $m_x \subset \mathcal{O}_{X,x}$ as the set of functions that vanish at x. m_x / m_x^2 represents the set of functions that vanish at x to first order.

$x = (0, \dots, 0)$ in coordinates $x_1, \dots, x_n$

$$f = \underbrace{f(0)}_{=\,0} + \sum_{i=1}^{n} a_i x_i + \underbrace{\sum_{1 \le i \le j \le n} b_{ij} \cdot x_i x_j + \cdots}_{\in\, m_x^2}$$

We think of tangent vectors as dual to the functions,

so $\left(m_x / m_x^2 \right)^*$ is the right algebraic object

for the tangent space at $x \in X$.

With differentials, we globalize this.

Derivations are the algebraic analogues of "taking derivatives"

Def: Let A (= "constants") be a commutative ring, B a commutative A-algebra (= "functions"), M a B-module (= "where the derivatives live").

An A-derivation of B into M is an A-linear map

$\Downarrow$

($da = 0 \ \forall a \in A$ follows from Leibnitz rule)

$$d : B \longrightarrow M$$

s.t. (Leibnitz rule) $\forall\, b, b' \in B$

$$d(bb') = b\,d(b') + b'\,d(b)$$

The set $Der_A(B, M)$ of all A-derivations of B into M has a natural structure of B-module defined by $(bd)(b') = b(d(b'))$ $\forall\, b, b' \in B$, $d \in Der_A(B, M)$

We have a (covariant) functor

$$\underline{B\text{-modules}} \longrightarrow \underline{B\text{-modules}}$$

$$M \longmapsto \mathrm{Der}_A(B, M)$$

__Proposition and definition:__

There exists a B-module $\Omega^1_{B/A}$ with an A-derivation $d: B \longrightarrow \Omega^1_{B/A}$ s.t. $\forall$ B-module M with A-derivation $D: B \to M$, $\exists!$ hom. of B-modules $h: \Omega^1_{B/A} \longrightarrow M$ s.t. the following diagram is commutative

$$\begin{tikzcd} B \arrow[r, "d"] \arrow[dr, "D"'] & \Omega^1_{B/A} \arrow[d, "\exists! h"] \arrow[dl, phantom, "\circlearrowleft"] \\ & M \end{tikzcd}$$

The universal property implies that $(\Omega^1_{B/A}, d)$ is unique up to an isom. of B-mod.

It is called the module of relative differentials or differential forms of B over A.

<u>Proof</u>: We prove the existence by constructing $\Omega^1_{B/A}$ (in two different ways).

(1) $F :=$ free B-module with basis $\{db \mid b \in B\}$.

$\Omega^1_{B/A} :=$ quotient of F by the submodule generated by
$$\left\{ d(ab + a'b') - a\,db - a'\,db',\ d(bb') - b\,db' - b'\,db \;\middle|\; \begin{array}{l} \forall a, a' \in A \\ b, b' \in B \end{array} \right\}.$$

n $\Omega^1_{B/A}$ = quotient of F by the submodule generated by

$$\{da,\ d(bb')-bdb'-b'db \mid \forall a\in A, b,b'\in B\}$$

The differential $d: B\to\Omega^1_{B/A}$ is induced by

$$B\longrightarrow F,\quad b\longmapsto db.$$

(2) Let $m: B\otimes_A B\longrightarrow B$ be the multiplication map

$$b\otimes b'\longmapsto bb'$$

Let $I := \langle 1\otimes b - b\otimes 1,\ b\in B\rangle \subset B\otimes_A B$

be the kernel of m. Then $\Omega^1_{B/A} = I/I^2$ and

the differential $d: B\longrightarrow \Omega^1_{B/A} = I/I^2$ is

$$b\longmapsto \overline{1\otimes b - b\otimes 1} =: db$$

Idea: Put $Y = \operatorname{Spec} B$, $X = \operatorname{Spec} A$

$$Y \times_X Y = \operatorname{Spec} B \otimes_A B \longleftarrow Y = \operatorname{Spec} B$$

Δ = diagonal

$$\Longleftrightarrow \qquad B \otimes_A B \longrightarrow B$$

$$\begin{tikzcd}
Y \arrow[dr, "\Delta"] \arrow[drr, bend left, "Id_Y"] \arrow[ddr, bend right, "Id_Y"'] & & \\
& Y \times_X Y \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[dr, phantom, "\sqcap"] & Y \arrow[d] \\
& Y \arrow[r] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
B & & \\
& B \otimes_A B \arrow[ul, "m"'] & B \arrow[l, "b \mapsto b\otimes 1"'] \arrow[ull, bend right, "Id_B"'] \\
& B \arrow[u, "b \mapsto 1\otimes b"] \arrow[uul, bend left, "Id_B"] \arrow[ur, phantom, "\sqcap"] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

I = ideal of the image of the diagonal

I is a $(B \otimes_A B)$-module $\Rightarrow$ $I/I^2 = I \otimes_{B \otimes_A B} \left(B \otimes_A B / I \right)$ is a

$$\left(B \otimes_A B / I = B \right) \text{— module.}$$

Can verify that at a point $y \in Y$,

$$\left(I/I^2\right)_y \Big/ m_y \left(I/I^2\right)_y \cong m_y / m_y^2$$

<u>Example:</u> $A = k$ a field $B = k[x_1, \ldots, x_n]$

$P \in k[x_1, \ldots, x_n]$ $\quad dP = \sum_{i=1}^{n} \frac{\partial P}{\partial x_i} dx_i \in \Omega^1_{B/A}$

by the relations $da = 0$ and $d(bb') = b\,db' + b'\,db$

where $\frac{\partial P}{\partial x_i}$ = formal partial derivative with respect to x_i

e.g.: $P = x_1^2 + x_2^2 - x_3 x_4 \qquad \frac{\partial P}{\partial x_1} = 2x_1$

$$\Rightarrow \quad \Omega^1_{B/A} = B\,dx_1 \oplus \cdots \oplus B\,dx_n$$

Properties: (1) Base change: A' an A-algebra

define $B' := B \underset{A}{\otimes} A'$. Then

$$\Omega^1_{B'/A'} = \Omega^1_{B/A} \underset{B}{\otimes} B'.$$

(2) Pull-back: $\operatorname{Spec} C \to \operatorname{Spec} B \to \operatorname{Spec} A$

or $C \leftarrow B \leftarrow A$

then there is a natural exact sequence

$$\Omega^1_{B/A} \underset{B}{\otimes} C \longrightarrow \Omega^1_{C/A} \longrightarrow \Omega^1_{C/B} \longrightarrow 0$$
$$dc \longmapsto dc$$

(3) Restriction to a closed subscheme:

$$A \longrightarrow B \twoheadrightarrow C = B/I \qquad \left(\text{can show } \Omega^1_{C/B} = 0\right)$$

$$I/I^2 \xrightarrow{\delta} \Omega^1_{B/A} \otimes_B C \longrightarrow \Omega^1_{C/A} \longrightarrow 0$$

$$db \otimes 1 \longmapsto db$$

where $\delta(\bar{b}) = db \otimes 1$ for $b \in I$, $\bar{b}$ = image of b in I/I^2

Example: $A = k$ field $\quad B = k[x_1, \dots, x_n]$

$I = \langle f_1, \dots, f_r \rangle \subset B. \qquad C = B/I$

$$\Omega^1_{B/k} = B\,dx_1 \oplus \cdots \oplus B\,dx_n$$

$$\Rightarrow \quad \Omega^1_{B/k} \otimes_B C = C\,dx_1 \oplus \cdots \oplus C\,dx_n$$

$$\delta(\overline{f_i}) = df_i \otimes 1$$

$$df_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} dx_j$$

$$\Rightarrow \quad \Omega^1_{C/k} = \frac{C dx_1 \oplus \cdots \oplus C dx_n}{\langle df_1, \ldots, df_n \rangle}$$

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e.g.: (1) $C = k[x,y]/_{y-x^2}$ parabola

$$\Omega^1_{C/k} = \frac{C dx \oplus C dy}{d(y-x^2)} \qquad d(y-x^2) = dy - 2x\,dx$$

$\cong C\,dx$ is free of rank 1 over C.

(2) the cuspidal cubic: $C = k[x,y]/_{(y^2-x^3)}$