$$\delta(\overline{f_i}) = df_i \otimes 1$$

$$df_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} dx_j$$

$$\Rightarrow \quad \Omega^1_{C/k} = \frac{C\,dx_1 \oplus \cdots \oplus C\,dx_n}{\langle df_1, \ldots, df_r \rangle}$$

---

e.g.: (1) $C = k[x,y]/_{y-x^2}$ parabola

$$\Omega^1_{C/k} = \frac{C\,dx \oplus C\,dy}{d(y-x^2)} \qquad d(y-x^2) = dy - 2x\,dx$$

$\cong C\,dx$ is free of rank 1 over C.

(2) the cuspidal cubic: $C = k[x,y]/_{(y^2-x^3)}$

$$\Omega^1_{C/k} = \frac{C\,dx \oplus C\,dy}{d(y^2-x^3)} \qquad d(y^2-x^3) = 2y\,dy - 3x^2\,dx$$

char $\neq 2, 3$

$$= (C\,dx \oplus C\,dy) \big/ (2y\,dy - 3x^2\,dx)$$

On the open set $D(x)$;

$$C[x^{-1}] = \left(k[x,y] \big/ (y^2-x^3) \right) [x^{-1}]$$

By base change $\Omega^1_{C[x^{-1}]/k[x^{-1}]} = \Omega^1_{C[x^{-1}]/k}$

$$= \Omega^1_{C/k} \otimes_C C[x^{-1}] = \frac{C[x^{-1}]\,dx \oplus C[x^{-1}]\,dy}{2y\,dy - 3x^2\,dx}$$

x is invertible : $dx = \frac{2y}{3x^2}\,dy$ $\qquad \cong C[x^{-1}]\,dy$

free of rank 1.

Similarly, on $D(y)$ $\Omega^1_{C[y^{-1}]/k}$ is free of rank 1.

$D(x) \cup D(y) =$ cubic minus the origin

"$\Omega^1_{\text{cuspidal cubic}/k}$ is locally free away from $(0,0)$".

<u>Sheafification:</u> Suppose given a scheme X over a scheme S.

Cover S with open affine schemes $\operatorname{Spec} A$ and X with open affine schemes $\operatorname{Spec} B$ s.t. $\operatorname{Spec} B$ maps to $\operatorname{Spec} A$.

For each $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$, we define

$$\Omega^1_{\operatorname{Spec} B/\operatorname{Spec} A} := \widetilde{\Omega^1_{B/A}} \quad \text{quasi-coherent on } \operatorname{Spec} B$$

These glue to a global quasi-coherent sheaf $\Omega^1_{X/S}$

because differentials commute with localization by the base change property: $\begin{array}{c} A \longrightarrow B \\ \cup \\ S \end{array}$ $\quad S^{-1}B = B \otimes_A S^{-1}A$

$$\Omega^1_{S^{-1}B/S^{-1}A} = \Omega^1_{B/A} \otimes_B S^{-1}B$$

We saw (last quarter) that any intersection $\operatorname{Spec} A \cap \operatorname{Spec} A'$ can be covered with affine open sets that are basic for $\operatorname{Spec} A$ and $\operatorname{Spec} A'$, i.e., of the form

$$\operatorname{Spec} A[f^{-1}] = \operatorname{Spec} A'[f'^{-1}]$$

$$\Rightarrow \quad \operatorname{Spec} B[f^{-1}] = \operatorname{Spec} B'[f'^{-1}]$$

$$\Rightarrow \quad \Omega^1_{B/A}[f^{-1}] = \Omega^1_{B[f^{-1}]/A[f^{-1}]} = \Omega^1_{B'[f'^{-1}]/A'[f'^{-1}]} = \Omega^1_{B'/A'}[f'^{-1}]$$

<u>Remark</u>: One can also define $\Omega^1_{X/S}$ using the diagonal embedding. $\pi: X \longrightarrow S$

$$S = \bigcup_{j\in J} \operatorname{Spec} A_j \qquad \pi^{-1}(\operatorname{Spec} A_j) = \bigcup_{k\in K_j} \operatorname{Spec} B_{jk}$$

$$X \times_S X = \bigcup_{\substack{j\in J \\ k, k'\in K_j}} \operatorname{Spec} B_{jk} \otimes_{A_j} B_{jk'} \supset \bigcup_{\substack{j\in J \\ k\in K_j}} \operatorname{Spec} B_{jk} \otimes_{A_j} B_{jk}$$

open embedding

$$\Delta : X \xhookrightarrow[\text{closed embedding}]{} \bigcup_{\substack{j\in J \\ k\in K_j}} \operatorname{Spec} B_{jk} \otimes_{A_j} B_{jk} \xhookrightarrow[\text{open embedding}]{} X \times_S X$$

Let $\mathcal{J}$ be the sheaf of ideals of $\Delta(X)$ in $\bigcup_{jk} \operatorname{Spec} B_{jk} \otimes_{A_j} B_{jk}$

$$\Omega^1_{X/S} := \Delta^*\left(\mathcal{J}/\mathcal{J}^2\right)$$

We have a differential $d: \mathcal{O}_X \longrightarrow \Omega^1_{X/S}$
obtained from gluing the local differentials $B \to \Omega^1_{B/A}$.

<u>Properties</u>: (1) Base change:

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & A' \arrow[d] \\ B \arrow[r] & B' = B \otimes_A A' \end{tikzcd}$$

$$\begin{tikzcd} \operatorname{Spec} B' = X' = X \times_S S' \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\square"] & X = \operatorname{Spec} B \arrow[d] \\ \operatorname{Spec} A' = S' \arrow[r] & S = \operatorname{Spec} A \end{tikzcd}$$

$$\Omega^1_{B'/A'} = \Omega^1_{B/A} \otimes_B B'$$

$$\Rightarrow \quad \Omega^1_{\operatorname{Spec} B'/\operatorname{Spec} A'} = \widetilde{\Omega^1_{B'/A'}} = \widetilde{\Omega^1_{B/A} \otimes_B B'}$$

$$= p_1^* \Omega^1_{\operatorname{Spec} B/\operatorname{Spec} A}$$

In general ($\pi: X \to S$) these glue to $\Omega^1_{X'/S'} = p_1^* \Omega^1_{X/S}$
$(S' \to S \quad X' := X \times_S S')$

(2) Pull-back: $A \to B \to C$

$$\operatorname{Spec} C \xrightarrow[\rho]{} \operatorname{Spec} B \to \operatorname{Spec} A$$

$$\Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A} \to \Omega^1_{C/B} \to 0$$

$$\Rightarrow \quad \widetilde{\Omega^1_{B/A} \otimes C} \to \widetilde{\Omega^1_{C/A}} \to \widetilde{\Omega^1_{C/B}} \to 0$$

$$\shortparallel$$

$$\rho^* \Omega^1_{\operatorname{Spec} B/\operatorname{Spec} A} \to \Omega^1_{\operatorname{Spec} C/\operatorname{Spec} A} \to \Omega^1_{\operatorname{Spec} C/\operatorname{Spec} B} \to 0$$

In the general case $Y \xrightarrow[\rho]{} X \xrightarrow[\pi]{} S$

these glue to $\rho^* \Omega^1_{X/S} \to \Omega^1_{Y/S} \to \Omega^1_{Y/X} \to 0$

(3) Restriction to closed subschemes:

$$A \to B \to C = B/I$$

$$Y = \operatorname{Spec} C \xhookrightarrow[i]{} \operatorname{Spec} B = X \xrightarrow[\pi]{} \operatorname{Spec} A = S$$

$$I/I^2 \longrightarrow \Omega^1_{B/A} \otimes_B C \longrightarrow \Omega^1_{C/A} \longrightarrow 0$$

$$\Rightarrow \quad \widetilde{I/I^2} \longrightarrow \widetilde{\Omega^1_{B/A} \otimes_B C} \longrightarrow \widetilde{\Omega^1_{C/A}} \longrightarrow 0$$

$$\begin{array}{ccccc} \| & & \| & & \\ \mathcal{J}_Y/\mathcal{J}_Y^2 & \longrightarrow & i^*\Omega^1_{X/S} & \longrightarrow & \Omega^1_{Y/S} \longrightarrow 0 \end{array}$$

this is also what we get in general.

<u>Note</u>: In differential geometry, if we have a submanifold $Y \hookrightarrow X$, we have $0 \to T_Y \to T_X|_Y \longrightarrow N_{Y/X} \to 0$

this is the dual of the sequence above.

Varieties: Definition: A variety over a field k is an integral separated scheme of finite type over k.

Definition: A variety$/_k$ is called nonsingular if all its local rings are regular local rings.

Recall that a noetherian local ring R with maximal ideal m is regular if $\dim_k m/m^2 = \dim R$ where $k = R/m$ is the residue field. For general noetherian rings we have $\dim_k m/m^2 \geqslant \dim R$.

Note then that by Nakayama's lemma, the lifts of any basis of m/m^2 generate m. In the case $\dim R = 1$, regular local rings are DVRs..

__Remark:__ Any localization of a regular local ring is again regular. In other words, being regular is stable under generization.

__Def:__ The set of points of a scheme whose rings are NOT regular is called its singular locus.
The complement is the nonsingular (or regular) locus.

So the nonsingular locus is closed under generization and the singular locus is closed under specialization.
We shall see that when X is a variety over a field, its singular locus is a proper closed subset.