

Definition: (1) Given an abelian category  $\mathcal{A}$ , the category

$\mathcal{C}(\mathcal{A})$  is the category whose objects are complexes

$$A^\bullet = (\dots, A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \xrightarrow{d^{n+1}} \dots)$$

of objects and morphisms of  $\mathcal{A}$  where  $\forall n \in \mathbb{Z} \quad d^{n+1}d^n = 0$ , the morphisms in  $\mathcal{C}(\mathcal{A})$  are morphisms of complexes, i.e.,

$\varphi^\bullet : A^\bullet \rightarrow B^\bullet$  is the data of a collection of

morphisms  $\varphi^n : A^n \rightarrow B^n \quad \forall n$  s.t.  $\varphi^{n+1}d^n = d^n\varphi^n$ :

$$\begin{array}{ccc}
 A^{n-1} & \xrightarrow{\quad} & B^{n-1} \\
 d^{n-1} \downarrow & \varphi^{n-1} \curvearrowright & \downarrow d^{n-1} \\
 A^n & \xrightarrow{\quad} & B^n \\
 & \varphi_n \curvearrowright & \\
 d^n \downarrow & \varphi^n \curvearrowright & \downarrow d^n \\
 A^{n+1} & \xrightarrow{\quad} & B^{n+1} \\
 & \varphi_{n+1} \curvearrowright & 
 \end{array}$$

(2) A homotopy between two morphisms  $\varphi, \psi: A \rightarrow B$  is a collection of maps  $h^n: A^n \rightarrow B^{n-1}$  s.t.

$$\forall n \quad \psi^n - \varphi^n = d^{n-1} h^n + h^{n+1} d^n$$

$$\begin{array}{ccc}
 A^{n-1} & \longrightarrow & B^{n-1} \\
 d^{n-1} \downarrow & \nearrow h^n & \downarrow d^{n-1} \\
 A^n & \xrightarrow{\psi^n - \varphi^n} & B^n \\
 d^n \downarrow & \nearrow h^{n+1} & \downarrow d^n \\
 A^{n+1} & \longrightarrow & B^{n+1}
 \end{array}$$

A morphism is called null-homotopic if it is homotopic to the zero morphism ( $\forall A, B \in \mathcal{A}$   $A \rightarrow 0 \rightarrow B$  is the zero homomorphism).

(3) The homotopy category  $\mathcal{H}b(\mathcal{A})$  is the category with the same objects as  $\mathcal{C}(\mathcal{A})$ , and whose morphisms are the homotopy classes of morphisms of complexes.

(homotopy defines an equivalence relation on morphisms in  $\mathcal{C}(\mathcal{A})$ ,  $\text{Hom}_{\mathcal{H}b(\mathcal{A})}(A^\cdot, B^\cdot) = \text{Hom}_{\mathcal{C}(\mathcal{A})}(A^\cdot, B^\cdot) / \text{homotopy}$ )

(4) Given a complex  $A^\cdot \in \text{Ob}(\mathcal{C}(\mathcal{A}))$ , the  $i$ -th cohomology object  $h^i(A^\cdot)$  is the quotient

$$h^i(A^\cdot) := \text{Ker } d^i / \text{Im } d^{i-1}$$

Any morphism of complexes  $\varphi^\cdot: A^\cdot \rightarrow B^\cdot$  induces

morphisms  $h^i(\varphi^\cdot): h^i(A^\cdot) \rightarrow h^i(B^\cdot)$  obtained

from the commutativity of

$$\begin{array}{ccc} A^{n-1} & \xrightarrow{\varphi^{n-1}} & B^{n-1} \\ d^{n-1} \downarrow & & \downarrow d^{n-1} \\ A^n & \xrightarrow{\varphi^n} & B^n \\ d^n \downarrow & & \downarrow d^n \\ A^{n+1} & \xrightarrow{\varphi^{n+1}} & B^{n+1} \end{array}$$

Given an exact sequence of complexes:

$$0 \rightarrow A^\cdot \rightarrow B^\cdot \rightarrow C^\cdot \rightarrow 0$$

there is a sequence of coboundary morphisms

$\delta^i : h^i(C) \rightarrow h^{i+1}(A)$  such that the  
 sequence

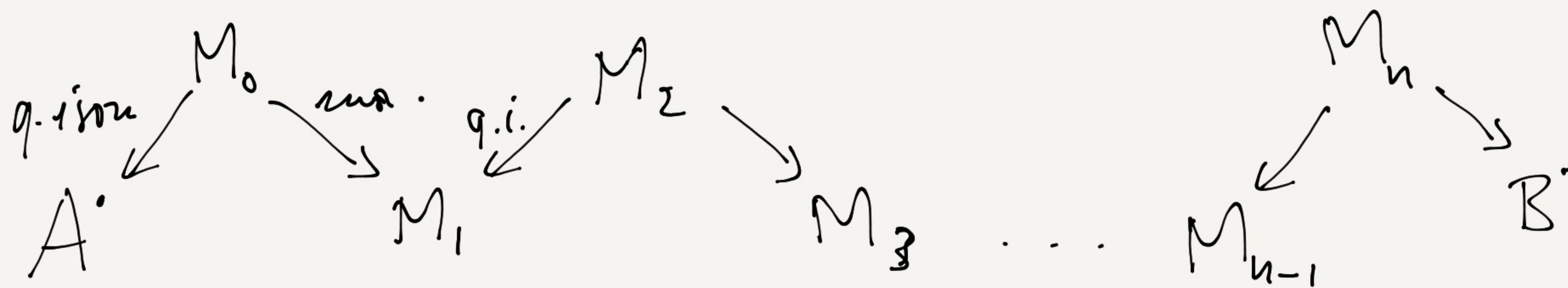
$$\dots \rightarrow h^i(A) \rightarrow h^i(B) \rightarrow h^i(C) \xrightarrow{\delta^i} h^{i+1}(A) \rightarrow \dots$$

is exact.

A morphism of complexes is a quasi-isomorphism if it induces isomorphisms in cohomology.

(5) The derived category  $\mathcal{D}(\mathcal{A})$  of  $\mathcal{A}$  is the "localization" of  $\mathcal{H}(\mathcal{A})$  at the "multiplicative set" of quasi-isomorphisms.

This means that a morphism  $\underline{\Phi} : A \rightarrow B$  in  $\mathcal{D}(\mathcal{A})$  is a finite sequence



Hence two complexes gives isomorphic objects of  $\mathcal{D}(\mathcal{A})$  if and only if they can be connected by a chain of quasi-isomorphisms.

The category  $\mathcal{A}$  embeds in  $\mathcal{C}(\mathcal{A}), \mathcal{H}(\mathcal{A}), \mathcal{D}(\mathcal{A})$  by sending an object  $A$  to the complex

$$\dots \rightarrow 0 \rightarrow A = A^0 \rightarrow 0 \rightarrow 0 \rightarrow \dots$$

(6) An injective resolution of  $A \in \mathcal{G}(\mathcal{A})$  is a complex  $I^\bullet$  of injective objects which is 0 in negative degree, together with a quasi-isomorphism  $\varepsilon: A \rightarrow I^\bullet$

This means that the sequence

$$0 \rightarrow A \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \dots$$

is exact.

$$\begin{array}{ccccccc}
 0 & \rightarrow & 0 & \rightarrow & I^0 & \rightarrow & I^1 \rightarrow I^2 \rightarrow \dots \\
 & & & & \uparrow & & \\
 0 & \rightarrow & 0 & \rightarrow & A & \rightarrow & 0 \rightarrow 0 \rightarrow \dots
 \end{array}$$

A projective resolution is the dual notion (reverse all the arrows).

If  $\mathcal{A}$  has enough injectives, then every object has an injective resolution.

Choose  $I^0$  s.t.  $0 \rightarrow A \rightarrow I^0$

Choose  $I^1$  s.t.  $I^0/A \hookrightarrow I^1, I^0 \rightarrow I^0/A \rightarrow I^1$

"  $I^2$  s.t.  $I^1 / \text{image of } I^0 \hookrightarrow I^2 \dots$

Dually, if  $\mathcal{A}$  has enough projectives, then every object has a projective resolution.

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Derived functors:

Suppose  $\mathcal{A}$  has enough injectives and let  $F: \mathcal{A} \rightarrow \mathcal{B}$  be a covariant left exact functor to another abelian category.

$F$  is left exact if,  $F$  is additive, i.e;

$$F_{A,B}: \text{Hom}_{\mathcal{A}}(A, B) \rightarrow \text{Hom}_{\mathcal{B}}(F(A), F(B))$$

is a hom. of abelian groups.

and  $\forall$  exact sequences  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$   
 $0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C)$  is exact.



The right derived functors  $R^i F$  of  $F$  can be defined as  $R^i F(A) := H^i(F(I^\bullet))$  where  $I^\bullet$  is an injective resolution of  $A$ .

Lemma 1: This is well-defined, i.e., independent of the choice of injective resolution.

Follows from:

Lemma 2:  $\forall$  injective resolution  $I^\bullet$  of an object  $B$  and an arbitrary resolution  $C^\bullet$  of an object  $A$ , and any morphism  $u: A \rightarrow B$ ,  $\exists v^\bullet: C^\bullet \rightarrow I^\bullet$  s.t.

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ \downarrow & \mathcal{Q} & \downarrow \\ C^\bullet & \xrightarrow{v^\bullet} & I^\bullet \end{array}$$

commutes.

Furthermore,  $r^i$  is unique up to homotopy.

Proof:

$$\begin{array}{ccccccccccc}
 \rightarrow & 0 & \rightarrow & A & \rightarrow & C^0 & \rightarrow & C^1 & \rightarrow & C^2 & \rightarrow & \dots \\
 & \downarrow 0 & & \downarrow 0 & & \downarrow \mu & & & & & & \\
 \dots & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & B & \rightarrow & I^0 & \rightarrow & I^1 & \rightarrow & I^2 & \rightarrow & \dots
 \end{array}$$

$$r^{-1} := \mu$$

Suppose we have constructed  $r^i$  for  $i \leq m-1$

$$\begin{array}{ccccc}
 C^{m-2} & \xrightarrow{d} & C^{m-1} & \xrightarrow{d} & C^m \\
 \downarrow r^{m-2} & \wr & \downarrow r^{m-1} & \wr & \downarrow r^m \\
 I^{m-2} & \xrightarrow{d} & I^{m-1} & \xrightarrow{d} & I^m
 \end{array}$$

$d^{m-1} r^{m-1} d^{m-2} = d^{m-1} d^{m-2} r^{m-2} = 0$  so  $d^{m-1} r^{m-1}$  is zero on  $\text{im } d^{m-2} = \ker d^{m-1}$ , i.e., factors through  $C^{m-1} / \ker d^{m-1} \hookrightarrow C^m$ .  
 Since  $I^m$  is injective, this factorization, hence also  $d^{m-1} r^{m-1}$  extends to  $r^m: C^m \rightarrow I^m$ .

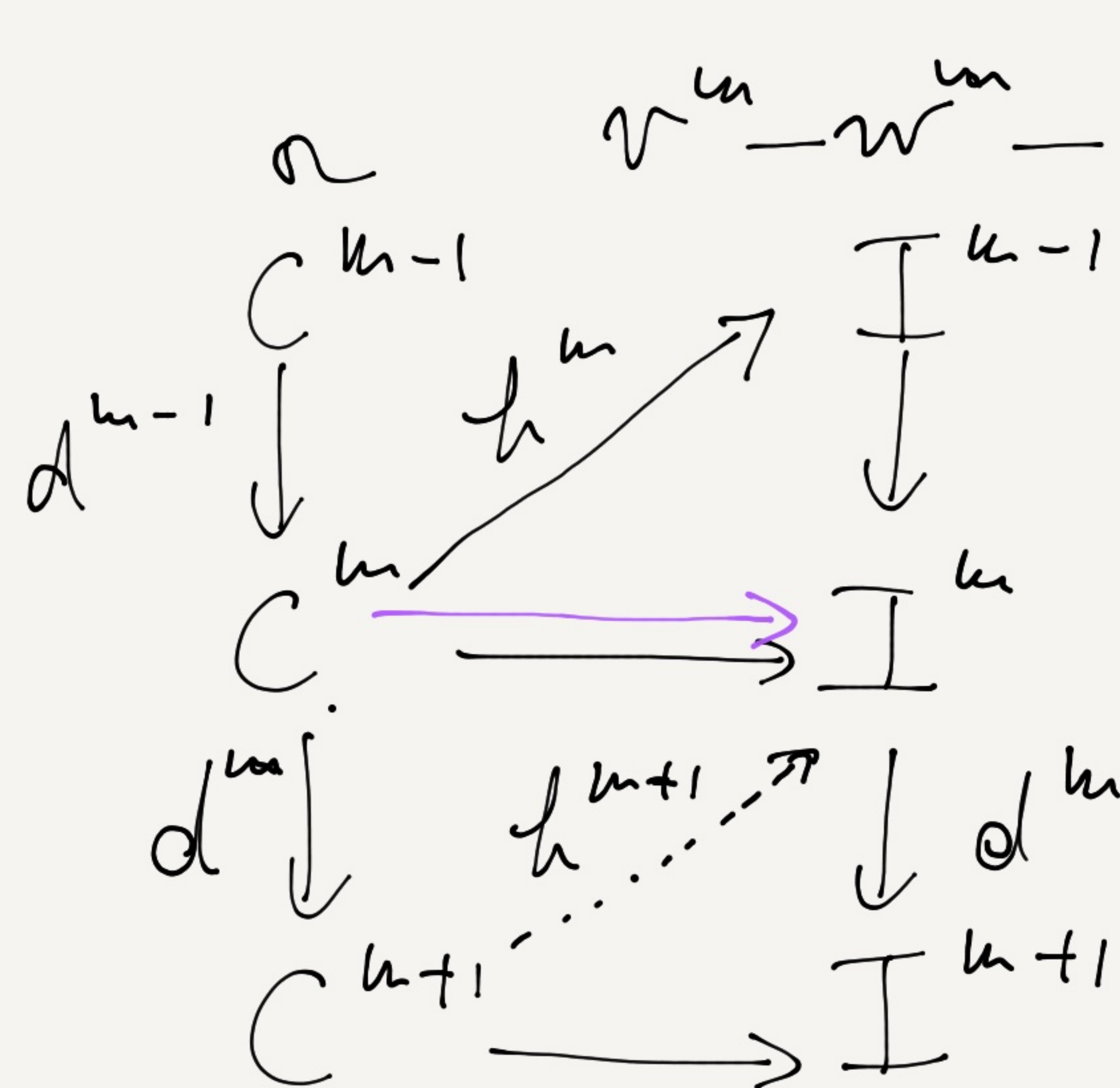
Given two morphisms of complexes  $v, w : C \rightarrow I$  which are both extensions of  $u$ , we show that they are homotopic.

We put  $h^{-1} = 0 : A = C^{-1} \rightarrow 0$

Suppose we have constructed  $h^i$  for  $i \leq n$  :  $C^i \rightarrow I^{i-1}$

s.t.  $v^i - w^i = dh^i + h^{i+1}d^i$

we want  $h^{n+1}$  s.t.  $v^n - w^n = dh^n + h^{n+1}d^n$



to show the existence of  $h^{n+1}$ , we need to show that  $v^n - w^n - d^{n-1}h^n$  factors through  $C^{n+1}$  (a  $d^n$ )

$v^m - w^m - dh^m$  is zero on  $\text{im } d^{m-1}$  because

$$\begin{aligned} (v^m - w^m - dh^m) d &= (v^m - w^m) d - d(h^m d) \\ &= (v^m - w^m) d - d(v^{m-1} - w^{m-1} - dh^{m-1}) \\ &= 0 \end{aligned}$$

So  $v^m - w^m - dh^m$  is zero on  $\ker d^m = \text{im } d^{m-1}$  and factors

through  $\begin{array}{ccc} C^m & & C^{m+1} \\ \text{ker } d^m & \hookrightarrow & \end{array}$ .

This factorization and hence also  $v^m - w^m - dh^m$  extends to a morphism  $h^{m+1}: C^{m+1} \rightarrow I^m$ .

□