It remains to prove that $f_1, \dots, f_r$ generate the unit ideal in $H^0(X, \mathcal{O}_X)$.

This means that the map

$$H^0(X, \mathcal{O}_X)^{\oplus r} \longrightarrow H^0(X, \mathcal{O}_X)$$
$$(a_1, \dots, a_r) \longmapsto \sum_{i=1}^{r} a_i f_i$$

is surjective

This is the map on global sections of the morphism of sheaves:

$$\mathcal{O}_X^{\oplus r} \longrightarrow \mathcal{O}_X$$

on any U $\quad (b_1, \dots, b_r) \longmapsto \sum_{i=1}^{r} b_i f_i \qquad b_i \in \mathcal{O}_X(U)$

This is a surjective morphism of sheaves because $X = \bigcup_{i=1}^{r} X_{f_i} \Rightarrow \forall x \in X, \exists i$ s.t. $(f_i)_x \notin m_x$

i.e., $(f_i)_x$ generates $\mathcal{O}_{X,x}$.

Let $\mathcal{F}$ be the kernel of this morphism, so that we have the exact sequence:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X^{\oplus r} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Using the long exact sequence of cohomology, if $H^1(X, \mathcal{F}) = 0$, then $H^0(\mathcal{O}_X^{\oplus r}) \longrightarrow H^0(\mathcal{O}_X)$ is surjective.

<u>Filter</u>:

$$\begin{array}{ccccccccc}
\mathcal{O}_X & \subset & \mathcal{O}_X^{\oplus 2} & \subset & \mathcal{O}_X^{\oplus 3} & \subset & \cdots & \subset & \mathcal{O}_X^{\oplus r} \\
\cup & & \cup & & \cup & & & & \cup \\
\mathcal{F} \cap \mathcal{O}_X & \subset & \mathcal{F} \cap \mathcal{O}_X^{\oplus 2} & \subset & \mathcal{F} \cap \mathcal{O}_X^{\oplus 3} & \subset & \cdots & \subset & \mathcal{F}
\end{array}$$

<u>First</u>:

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{F} \cap \mathcal{O}_X & \longrightarrow & \mathcal{F} \cap \mathcal{O}_X^{\oplus 2} & \longrightarrow & \mathcal{F} \cap \mathcal{O}_X^{\oplus 2} / \mathcal{F} \cap \mathcal{O}_X & \longrightarrow & 0 \\
 & & \cap & & \cap & & \cap & & \\
0 & \to & \mathcal{O}_X & \longrightarrow & \mathcal{O}_X^{\oplus 2} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0
\end{array}$$

__Čech cohomology__: sometimes a practical way of computing coherent cohomology

__The general set-up:__

X a topological space

$\mathcal{F}$ a sheaf of abelian groups on X.

$\mathcal{U} = \{U_i \mid i \in I\}$ an open covering of X.

The axiom of choice implies that any set has at least one well-ordering (a total ordering s.t. every non-empty subset has a minimal element).

Choose a well-ordering on I.

For any finite subset $\{i_0 < i_1 < \cdots < i_n\} \subset I$, let $U_{i_0,\dots,i_r}$ be the intersection $U_{i_0} \cap \cdots \cap U_{i_r}$.

with $u_{i_0 \cdots i_r} : U_{i_0 \cdots i_r} \hookrightarrow X$

and put $\mathcal{F}_{i_0 \cdots i_r} := (u_{i_0 \cdots i_r})_* \left(\mathcal{F} |_{U_{i_0 \cdots i_r}} \right)$.

Define coboundary maps:

$$d^r : \prod_{i_0 < \cdots < i_r \in I} \mathcal{F}_{i_0 \cdots i_r} \longrightarrow \prod_{i_0 < \cdots < i_{r+1}} \mathcal{F}_{i_0,\dots,i_{r+1}}$$

$$\begin{array}{ccc} \cup\!\!| & & \Psi \\ \alpha = \left(\alpha_{i_0 \cdots i_r} \right) & \longmapsto & d\alpha \end{array}$$

$$\left(d\alpha \right)_{i_0 \cdots i_{r+1}} = \sum_{p=0}^{r+1} (-1)^p \alpha_{i_0 \cdots \hat{i_p} \cdots i_{r+1}} \Big|_{U_{i_0 \cdots i_{r+1}}}$$

The coboundaries form the long exact sequence of sheaves:

$$0 \longrightarrow \mathcal{F} \longrightarrow \prod_{i_0 \in I} \mathcal{F}_{i_0} \xrightarrow{d^0} \prod_{i_0 < i_1 \in I} \mathcal{F}_{i_0 i_1} \xrightarrow{d^1} \prod_{i_0 < i_1 < i_2 \in I} \mathcal{F}_{i_0 i_1 i_2} \xrightarrow{d^2} \cdots$$

The exactness is left as an exercise. For instance, the fact that the kernel of d^0 is the image of $\mathcal{F}$ is equivalent to the sheaf axioms for $\mathcal{F}$. The above sequence is called the Čech resolution of $\mathcal{F}$.

<u>Definition:</u> The Čech cohomology $\check{H}^{\bullet}(\mathcal{U}, \mathcal{F})$ of $\mathcal{F}$ for the open covering $\mathcal{U}$ is the cohomology of the complex of global sections of the exact sequence of sheaves above. In general, this would depend

on the choice of the well-ordering of I.

We shall prove that for a noetherian separated scheme with an affine cover, the Čech cohomology groups of quasi-coherent sheaves are naturally isomorphic to the usual derived cohomology groups.

More notation: $\check{\mathcal{C}}^p(\mathcal{F}) := \prod_{i_0 < \cdots < i_p \in I} \mathcal{F}_{i_0 \cdots i_p}$.

$$C^p(\mathcal{F}) := \prod_{i_0 < \cdots < i_p \in I} \mathcal{F}(U_{i_0 \cdots i_p})$$

the global sections of $\check{\mathcal{C}}^p(\mathcal{F})$

Terminology: These are called the Čech p-cochains.

$$\check{H}^r(\mathcal{U}, \mathcal{F}) = H^r(C^\cdot(\mathcal{F}))$$ by definition.

We need some preliminary results, interesting in their own right:

<u>Lemma</u>: X top. space. If $\mathcal{F}$ is flasque, then so are the sheaves $\mathcal{C}^r(\mathcal{F})$.

<u>Proof</u>: The restriction of a flasque sheaf is flasque, direct images of flasque sheaves are flasque, products of flasque sheaves are flasque. $\square$.

<u>Lemma</u>: If $\mathcal{F}$ is flasque, then $\forall$ $\mathcal{U}$ and all $r > 0$, we have $\check{H}^r(\mathcal{U}, \mathcal{F}) = 0$.

<u>Proof</u>: We have the Čech resolution:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0(\mathcal{F}) \xrightarrow{d^0} \mathcal{C}^1(\mathcal{F}) \longrightarrow \cdots$$

unwind into short exact sequences.

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0 \xrightarrow{d^0} \mathcal{B}^1 \longrightarrow 0 \quad \Rightarrow \mathcal{B}^1 \text{ flasque}$$

$$0 \longrightarrow \mathcal{B}^1 \xrightarrow{d^1} \mathcal{C}^1 \xrightarrow{d^2} \mathcal{B}^2 \longrightarrow 0 \quad \Rightarrow \mathcal{B}^2 \text{ flasque}$$

$$\vdots \qquad\qquad \vdots$$

$$0 \longrightarrow \mathcal{B}^l \longrightarrow \mathcal{C}^l \longrightarrow \mathcal{B}^{l+1} \longrightarrow 0 \quad \Rightarrow \mathcal{B}^{l+1} \text{ flasque.}$$

$$\vdots$$

$\Rightarrow$ all sequences of global sections obtained from the above are exact.

$\Rightarrow$ the Čech complex of global sections is exact

$\Rightarrow$ $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0 \quad \forall p > 0$ $\square$.

Construction: X top. space.

Recall that for any resolution $\mathcal{F} \to \mathcal{R}^\cdot$ and any injective resolution $\mathcal{F} \to \mathcal{I}^\cdot$, $\exists$ a morphism $u^\cdot : \mathcal{R}^\cdot \to \mathcal{I}^\cdot$ extending $Id : \mathcal{F} \to \mathcal{F}$, i.e.,

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "Id"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{F} \arrow[d] \\
\mathcal{R}^\cdot \arrow[r, "u^\cdot"] & \mathcal{I}^\cdot
\end{tikzcd}$$

$u^\cdot$ unique up to homotopy.

Apply this to $\mathcal{F} \longrightarrow \mathcal{C}^\cdot(\mathcal{F})$.

$u^\cdot : \mathcal{C}^\cdot(\mathcal{F}) \longrightarrow \mathcal{I}^\cdot$ unique up to homotopy

passing to the complexes of global sections, defines

$\kappa : \check{H}^p(\mathcal{U}, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F})$ unique.

Theorem: X noetherian scheme, $\mathcal{F}$ quasi-coherent sheaf of $\mathcal{O}_X$-modules. $\mathcal{U}$ a covering of X by open affine schemes U_i, $i \in I$ s.t. $\forall$ $i_0 < .. < i_n \in I$, $U_{i_0 \cdots i_n}$ is affine. Then the canonical morphism

$$\kappa : \check{H}^n(\mathcal{U}, \mathcal{F}) \longrightarrow H^n(X, \mathcal{F})$$

is an isomorphism $\forall$ n.

Proof: Recall that $\mathcal{F}$ can be embedded in a flasque quasi-coherent sheaf, say $\mathcal{G}$. Let $\mathcal{R}$ be the quotient of $\mathcal{G}$ by $\mathcal{F}$, so that we have the exact sequence of quasi-coherent sheaves

$$0 \to \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{R} \longrightarrow 0$$