matrix with coefficients in A s.t. the height of any minimal prime of I is the expected codimension $(p-r+1)(q-r+1)$.

The main example of such an ideal is the ideal of the locus of matrices of rank $< r$ in the space of all $p \times q$ matrices with entries in a field.

Curves: From now on, all schemes are over an algebraically closed field k.

Def: A curve $X/_k$ is an integral separated scheme of finite type over k, of dimension 1. We say X is complete if it is proper over k. We will prove some nice results about curves.

Lemma 1: X nonsingular curve$/k$, $f: X \dashrightarrow Y$ a rational map (i.e., f is a morphism from an open dense subscheme of X to Y) to a projective variety Y.

Then f extends to a morphism $X \longrightarrow Y$.

Proof: Choose an embedding $Y \hookrightarrow \mathbb{P}^n_k$.

We can replace Y with $\mathbb{P}^n$ because if f extends to a morphism $X \longrightarrow \mathbb{P}^n$, then the morphism factors through Y because Y is a closed subscheme of $\mathbb{P}^n$ and X is integral.

Step 1: Let U be a nonempty open set where f is well-defined.

Lemma 2: The datum of a rational map $f: X \dashrightarrow \mathbb{P}^n$ is equivalent to the data of an invertible sheaf $\mathcal{L}$ on X and sections $s_0, \dots, s_n \in H^0(X, \mathcal{L})$ s.t. $\mathcal{L}|_U \cong f^*\mathcal{O}_{\mathbb{P}^n}(1)$,

$\forall i \;\; s_i|_U = f^*X_i$ and $U \subseteq \{x \in X \mid \exists i \text{ s.t. } (s_i)_x \notin m_x \mathcal{L}_x\}$

Proof: Put $\mathcal{M} := f^*\mathcal{O}_{\mathbb{P}^n}(1)$ which is an invertible sheaf on U,

and put $s_i := f^*X_i$ for $i = 0, \dots, n$.

Also put $\{P_1, \dots, P_r\} := X \setminus U$.

Via the exact sequence

$$\bigoplus_{i=1}^{r} \mathbb{Z}[P_i] \longrightarrow Cl(X) \xrightarrow[\text{restriction}]{} Cl(U) \longrightarrow 0$$

choose $\mathcal{L}''$ on X s.t. $\mathcal{L}''|_U = \mathcal{M}$.

Recall that $\mathcal{L}'' \otimes \mathcal{K}_X \cong \mathcal{K}_X$ because both sheaves are the constant sheaf with group $K(X)$.

Definition: A rational section of an invertible sheaf $\mathcal{L}$ on an integral scheme X is a global section s of $\mathcal{K}_X$ s.t.

$\exists\ V \neq \phi$, V open $\subset X$ with $s|_V \in H^0(X, \mathcal{L})$ via the embedding $\mathcal{L} \hookrightarrow \mathcal{L} \otimes \mathcal{K}_X = \mathcal{K}_X$. We say s is regular on V.

Back to the proof of Lemma 2:

We have
$$\begin{array}{ccc} \mathcal{L}''|_U & \hookrightarrow & (\mathcal{K}_X = \mathcal{K}_X \otimes \mathcal{L}'')|_U \\ & & \parallel \\ \mathcal{M} & \hookrightarrow & \mathcal{K}_U = \mathcal{K}_U \otimes \mathcal{M} \end{array}$$

So $s_0, \ldots, s_n \in H^0(U, \mathcal{M}) \hookrightarrow H^0(\mathcal{K}_U) = K(X) = H^0(\mathcal{K}_X)$ are rational sections of $\mathcal{L}''$, regular on U.

For $i = 1, \ldots, r$, put $m_i := \mathrm{Max}\{0, -v_{P_i}(s_j), j = 0, \ldots, n\}$

Then $\mathcal{O}_X(-m_1 P_1 - \cdots - m_r P_r) \hookrightarrow X$ (ideal sheaf)

and $\mathcal{L}'' \hookrightarrow \mathcal{L} := \mathcal{L}'' \otimes \mathcal{O}_X(m_1 P_1 + \cdots + m_r P_r)$

and $H^0(X, \mathcal{L}'') \hookrightarrow H^0(X, \mathcal{L}) \hookrightarrow H^0(X, \mathcal{K}_X) = K(X)$

at P_i: $\mathcal{L}_{P_i} = (\mathcal{L}'' \otimes \mathcal{O}_X(m_i P_i))_{P_i} = \pi_{P_i}^{m_i} \mathcal{L}''_{P_i}$ $\subset \mathcal{K}_{X, P_i} = K(X)$

π_{P_i} is a uniformizer at P_i

Ex: Show that $\forall\, s \in K(X)$, $s \in H^0(\mathcal{L}) \iff s_x \in \mathcal{L}_x \subset K(X)\ \forall x \in X$.
From this deduce that:
The sections of $\mathcal{L}$ are the rational sections of $\mathcal{L}''$ with poles of order at most n_i $\forall i$. (You can also deduce this by analyzing the transition functions for a suitable cover)
Hence $s_0, \ldots, s_n \in H^0(X, \mathcal{L})$.
By construction $U \subseteq \{x \in X \mid \exists\, i,\ (s_i)_x \notin m_x \mathcal{L}_x\}$.
$\square$

Proof of Lemma 1 continued:

Now put $n_j := \mathrm{Min}\{\nu_{P_j}((s_i)_{P_j}) : i = 0, \ldots, n\}$

and $\mathcal{L}' := \mathcal{L} \otimes \mathcal{O}_X(-n_1 P_1 - \cdots - n_r P_r) \hookrightarrow \mathcal{L}$

At P_i: $\mathcal{L}'_{P_i} = \pi_i^{n_i} \mathcal{L}_{P_i}$

And, as in the proof of Lemma 2, we can identify the sections of $\mathcal{L}'$ with the sections of $\mathcal{L}$ which vanish to order at least n_i at P_i.

locally, all s_j are divisible by $\pi_i^{n_i}$ $\forall i$, by the definition of the n_i

$\Rightarrow$ via the embedding of $H^0(X, \mathcal{L}') \hookrightarrow H^0(X, \mathcal{L})$ we can consider $s_0, \dots, s_n$ to be global sections of $\mathcal{L}'$ (they belong to the image of $H^0(X, \mathcal{L}')$)

Now map X to $\mathbb{P}^n$ via $\mathcal{L}'$ and $s_0, \dots, s_n \in H^0(X, \mathcal{L}')$

$$\rightsquigarrow \quad \varphi : X \to \mathbb{P}^n$$

$$\forall i \quad \exists j. \quad \text{s.t.} \quad (s_j)_{P_i} \notin m_{P_i} \cdot \underbrace{\mathcal{L}'_{P_i}}_{= \pi_i^{n_i} \mathcal{L}_{P_i}}$$

So φ is well-defined everywhere

and $\varphi|_U = f$ because $\mathcal{L}'|_U = \mathcal{L}|_U$ $\square$

<u>Remark</u> (Exercise). Similar arguments will prove:

If X is an integral, separated, locally factorial scheme of finite type$/_k$, the datum of a rational map $f : X \dashrightarrow \mathbb{P}^n_k$ is equivalent to the data of an invertible sheaf $\mathcal{L}$ and rational sections $s_0, \dots, s_n$ of $\mathcal{L}$ which generate $\mathcal{L}$ on a nonempty open set U s.t.

$$f^* \mathcal{O}_{\mathbb{P}^n}(1) \cong \mathcal{L}|_U \quad \text{and} \quad f^* X_i = s_i|_U \ .$$

Furthermore, the locus of indeterminacy of f, i.e., the complement of the largest open set to which f can be extended, has codimension ≥ 2.

Proposition 1: X a curve$/k$ (integral, separated, finite type, dim. 1)

Then X is complete if and only if it is projective.

Proof: Clearly, if X is projective, then it is complete.

Assume X is complete.

Cover X with open affine subsets $U_1, \dots, U_r$.

$\forall i$ we can write $U_i \cong \operatorname{Spec} A_i$ with

$$A_i = k[x_1, \dots, x_n]/I \implies U_i \hookrightarrow \mathbb{A}^n = \operatorname{Spec} k[x_1, \dots, x_n]$$

(we can assume one n works for all i)

embed $\mathbb{A}^n \hookrightarrow \mathbb{P}^n_k$ as one of the usual open affine subsets.

Let Y_i be the closure of the image of U_i in $\mathbb{P}^n$.

Note that Y_i (with its reduced induced structure) is an integral projective curve.

$$X \hookleftarrow U_i \hookrightarrow Y_i$$
$$\neq \emptyset$$

defines a rational map $X \dashrightarrow Y_i$ which, using Lemma 1 extends to a morphism $X \longrightarrow Y_i$.

Now show that the resulting product morphism

$$X \longrightarrow \prod_{i=1}^{r} Y_i$$ is a closed embedding (exercise)

This shows X is projective via the composition

$$X \hookrightarrow \prod_{i=1}^{r} Y_i \hookrightarrow \prod_{i=1}^{r} \mathbb{P}^{n} \xhookrightarrow[\text{Segre embedding}]{} \mathbb{P}^N$$

$\square$