HOMEWORK 8
Math100B UCSD Winter 2002
Due at the *start* of class on *Monday* 3/11

Reading assignment: Hoffman and Kunze, §§3.1–3.4, 6.2.

1. In class we saw a formula for the size of $\mathrm{GL}_n(\mathbf{F}_p)$ and $\mathrm{SL}_n(\mathbf{F}_p)$.

a) Compute the size of $\mathrm{GL}_n(\mathbf{F}_5)$ and $\mathrm{SL}_n(\mathbf{F}_5)$ for $n = 1, 2, 3, 4$.
b) Prove the group of strictly upper-triangular matrices in $\mathrm{GL}_n(\mathbf{F}_p)$ (1's on main diagonal, 0's below, arbitrary stuff above main diagonal) is a p-Sylow subgroup. Are these matrices also a p-Sylow subgroup of $\mathrm{SL}_n(\mathbf{F}_p)$?

2. This is a review exercise on inverse functions and assorted algebraic structures, to be sure you understand the basic idea here once and for all.

a) Let $f: X \to Y$ be a map of sets which is a bijection (that is, f is one-to-one and onto). Define the inverse function $f^{-1}: Y \to X$, by describing how the value $f^{-1}(y)$ is determined for $y \in Y$.

b) Let $f: G_1 \to G_2$ be a group homomorphism which is a bijection. Prove the inverse of f is a group homomorphism. That is, prove $f^{-1}(gg') = f^{-1}(g)f^{-1}(g')$ for all $g, g' \in G_2$.

c) Let $f: R_1 \to R_2$ be a ring homomorphism which is a bijection. Prove the inverse of f is a ring homomorphism. That is, $f^{-1}(1) = 1$, $f^{-1}(r + r') = f^{-1}(r) + f^{-1}(r')$, and $f^{-1}(rr') = f^{-1}(r)f^{-1}(r')$ for all $r, r' \in R_2$.

d) Let $f: V_1 \to V_2$ be a linear map of vector spaces over F which is a bijection. Prove the inverse of f is linear. That is, $f^{-1}(v + v') = f^{-1}(v) + f^{-1}(v')$ and $f^{-1}(cv) = cf^{-1}(v)$ for all $v, v' \in V_2$ and $c \in F$.

3. Let $\mathrm{Pol}_n(F) = F + FT + \cdots + FT^n$ be the span of the powers $1, T, \ldots, T^n$. One basis is $1, T, \ldots, T^n$, so $\mathrm{Pol}_n(F)$ has dimension $n + 1$. For example, $\mathrm{Pol}_2(F)$ is 3-dimensional with basis $\{1, T, T^2\}$.
Consider differentiation $D: \mathrm{Pol}_n(F) \to \mathrm{Pol}_n(F)$, which is a linear map. (Yes, the image is really in $\mathrm{Pol}_{n-1}(F)$, but think about it in $\mathrm{Pol}_n(F)$.)

a) Write $D: \mathrm{Pol}_2(F) \to \mathrm{Pol}_2(F)$, as a matrix relative to the basis $\{1, T, T^2\}$.

b) Show $\{T, T + 1, T^2 + T\}$ is also a basis of $\mathrm{Pol}_2(F)$, and write the matrix for differentiation on $\mathrm{Pol}_2(F)$ relative to this basis. Be careful! For matrix calculations, all results should be expressed in coordinates for this (unusual) basis, not for the standard basis $\{1, T, T^2\}$.

c) Compute the characteristic polynomials of both matrices in (a) and (b). (Hint: the two characteristic polynomials must be the same, so if they are not you'll know there was an error somewhere.)

4. For a polynomial $f(T) \in \mathbf{F}_2[T]$, set

$$A_f = \mathbf{F}_2[T]/(f(T)),$$

which is both a ring and an $\mathbf{F}_2$-vector space, of dimension $\deg f$. Let $\varphi_2: A_f \to A_f$ be the Frobenius map. That is, φ_2 is squaring on A_f, which is $\mathbf{F}_2$-linear.
Here are several polynomial choices for f:

$$T^2, \quad T^2 + 1, \quad T^2 + T + 1, \quad T^3 + 1, \quad T^3 + T + 1.$$

For each of these choices of f, do three things: factor f into irreducibles in $\mathbf{F}_2[T]$ (use problem 1 on set 6 to find the low degree irreducibles), compute the matrix of φ_2 on A_f relative to the basis of powers of T in A_f, and compute the characteristic polynomial of φ_2. Present your work cleanly.

5. (Composition and differentiation) The purpose of this final exercise is to illustrate that the linear algebra idea of thinking about numbers as scaling functions (that is, the number t is viewed as the function "scale by t") gives the chain rule of calculus a simple algebraic form.
Let $f: \mathbf{R} \to \mathbf{R}$ be a differentiable function. Define a new function $\mathrm{d}f: \mathbf{R}^2 \to \mathbf{R}^2$ by

$$(\mathrm{d}f)(x, y) = (f(x), f'(x)y).$$

The idea here is to think about the number $f'(x)$ as the function "scale by $f'(x)$"; the role of the second variable y is to make the scaling effect visible, since y is replaced with $f'(x)y$.

a) For the function $f(x) = x^3 - 5x + 1$, compute $(\mathrm{d}f)(2, 8)$.

b) Given two differentiable functions $f, g\colon \mathbf{R} \to \mathbf{R}$, we get two functions $\mathrm{d}f, \mathrm{d}g\colon \mathbf{R}^2 \to \mathbf{R}^2$. Compute the composite function $\mathrm{d}g \circ \mathrm{d}f\colon \mathbf{R}^2 \to \mathbf{R}^2$, and then explain why the chain rule of calculus is essentially equivalent to the nice formula

$$\mathrm{d}(g \circ f) = \mathrm{d}g \circ \mathrm{d}f.$$

Pretty neat!