HOMEWORK 5
Math 104B - Dr. Evans
UCSD Spring 2004
Due Thursday, May 13

1. Let $p > 2$ and let x be a p-adic number such that $x^p = 1$. We proved that x must be 1, as follows. If x is not 1, then x must have the form $x = 1 + tp^r$, where t is a p-adic unit and r is some positive integer. Thus $1 = x^p = (1 + tp^r)^p$. By the binomial theorem, the right side has the form $1 + tp^{r+1} + Mp^{2r+1}$, for some p-adic integer M. Bringing the first two of these terms over to the left, we get (after cancellation) $-t = Mp^r$, which contradicts the fact that t is a p-adic unit.

   For this problem, show precisely where this proof breaks down in the case $p = 2$. (*Hint: Look carefully at the binomial theorem.*)

2. Let $p = 13$. For every positive integer n, let $f(n)$ denote the number of distinct primitive p-adic n^{th} roots of unity. Find $f(1)$, $f(2)$, ..., $f(30)$. (*Hint:* $f(12) = 4$ *and* $f(15) = 0$.)

3. Problem #2 mentioned four primitive p-adic 12^{th} roots of unity, where $p = 13$. Give the first p-adic digit of each of these four roots of unity.

4. Let $p = 7$. Show that the two p-adic zeros of $f(x) = px^2 + x - 1$ have the following initial p-adic expansions:

$$x_1 = 1 + 6 \cdot 7 + 1 \cdot 7^2 + \dots$$
$$x_2 = 6 \cdot \frac{1}{7} + 5 + 5 \cdot 7^2 + \dots$$

   Guess the counterparts of these expansions when $p = 109$ in place of $p = 7$, using pattern recognition (no work need be shown).

5. Show that $x^4 - 17 = 2y^2$ has solutions x, y in $\mathbb{Q}_p$ for all p.

Reading Assignment

Read Section 3.5 about the Local-Global Principle. It's one of the main motivations for studying the p-adic numbers. If you were in Math 104A last quarter, you might want to compare the results in this section with Legendre's Theorem (see course web site from last quarter for details).