Midterm 1 Solutions
Math 104B - Dr. Evans
UCSD Spring 2004

1. Give the p-adic expansion of $\frac{121}{90}$, where $p = 5$.

$$\frac{121}{90} = \frac{1}{5}\left(\frac{121}{18}\right) = \frac{1}{5}\left(-\frac{5}{18} + 7\right)$$

Since $5^6 \equiv 1 \pmod{18}$, we have

$$-\frac{5}{18} = \frac{5\left(\frac{5^6-1}{18}\right)}{1-5^6} = \frac{4340}{1-p^6} = (3p + 3p^2 + 4p^3 + p^4 + p^5)(1 + p^6 + p^{12} + \dots)$$

Plugging this into the first expression,

$$\begin{aligned}\frac{121}{90} &= \frac{1}{p}\Big(\big(3p + 3p^2 + 4p^3 + p^4 + p^5 + 3p^7 + \dots\big) + \big(2 + p\big)\Big) \\ &= \frac{1}{p}\Big(2 + 4p + 3p^2 + 4p^3 + p^4 + p^5 + 3p^7 + \dots\Big) \\ &= \frac{2}{p} + 4 + 3p + 4p^2 + p^3 + p^4 + 3p^6 + \dots\end{aligned}$$

or in Evans' notation, $\frac{121}{90} = 2.4\overline{341103}$.

2. Prove that $x^3 = 10$ has a solution x in $\mathbb{Q}_3$.

Let $p = 3$. Let P_n be the following statement: "For some integer $x_n = a_0 + a_1 p + \dots + a_n p^n$, $x_n^3 \equiv 10 \pmod{3^{n+2}}$." When $n = 1$, the statement P_1 is true since $4^3 \equiv 10 \pmod{3^3}$.

Suppose now that the statement P_{n-1} holds true. Then there exists some integer $x_{n-1} = a_0 + a_1 p + \dots + a_{n-1}p^{n-1}$ such that $x_{n-1}^3 \equiv 10 \pmod{p^{n+1}}$, i.e., $10 - x_{n-1}^3 = kp^{n+1}$ for some integer k. Suppose $x_n = x_{n-1} + a_n p^n$ and assume $x_n^3 \equiv 10 \pmod{3^{n+2}}$. Then

$$\begin{aligned}10 \equiv x_n^3 &= \big(x_{n-1} + a_n p^n\big)^3 \equiv x_{n-1}^3 + 3x_{n-1}^2 a_n p^n \pmod{p^{n+2}} \\ kp^{n+1} &= 10 - x_{n-1}^3 \equiv x_{n-1}^2 a_n p^{n+1} \pmod{p^{n+2}} \\ k &= \frac{10 - x_{n-1}^3}{p^{n+1}} \equiv x_{n-1}^2 a_n \equiv x_0^2 a_n \equiv a_n \pmod{p} \\ a_n &\equiv \frac{10 - x_{n-1}^3}{p^{n+1}} \pmod{p}\end{aligned}$$

Since we were able to solve for a_n in terms of the previous term, x_n will satisfy $x_n^3 \equiv 10 \pmod{p^{n+2}}$. Hence, the statement P_n is true. We conclude that for all $n \geq 1$, $x_n^3 \equiv 10 \pmod{3^{n+2}}$ has an integer solution, where $x_n = a_0 + a_1 p + \dots + a_n p^n$. Since the x_n are "p-adically close," the sequence $\{x_n\}$ converges to a solution $x = a_0 + a_1 p + \dots$ in $\mathbb{Q}_3$.

3. Let $B(c,r) = \{\, x \in \mathbb{Q}_p \; : \; |x-c|_p < r \,\}$, where p is an odd prime. Prove that $B(\frac{3}{2},1)$ does not intersect $B(1,1)$.

Suppose $x \in B(\frac{3}{2},1) \cap B(1,1)$. Then $\left|x-\frac{3}{2}\right|_p < 1$ and $|x-1|_p < 1$. By the strong form of the triangle inequality for the p-adic absolute value,

$$\left|\frac{3}{2}-1\right|_p = \left|\left(\frac{3}{2}-x\right)+(x-1)\right|_p \leq \max\left\{\left|\frac{3}{2}-x\right|_p \, , \; |x-1|_p\right\} < 1$$

On the other hand, $\left|\frac{3}{2}-1\right|_p = \left|\frac{1}{2}\right|_p = 1$ for any odd prime p, since p divides neither the numerator nor the denominator. This is a contradiction, and so the intersection of the two balls must be empty.

4. True or False: $\sqrt{13}$ is in the field $\mathbb{Q}(e^{2\pi i/13})$ Justify very briefly.

In class, we showed that for odd primes p,

$$\mathcal{G}_p = \sum_{j=0}^{p+1} \zeta_p^{j^2} = \sum_{m=0}^{p-1} \left(\frac{m}{p}\right) \zeta_p^m = \begin{cases} \pm\sqrt{p} & \text{if } p \equiv 1 \pmod 4 \\ \pm i\sqrt{p} & \text{if } p \equiv 3 \pmod 4 \end{cases}$$

Since $13 \equiv 1 \pmod 4$ and $\mathcal{G}_{13} \in \mathbb{Q}(e^{2\pi i/13})$, we conclude that $\sqrt{13} \in \mathbb{Q}(e^{2\pi i/13})$, so the answer is **true**.