SOLUTIONS TO HOMEWORK 4
Math 104B - Dr. Evans
UCSD Spring 2004

1. Let $\{x_n\}$ be a sequence of p-adic numbers such that $x_n - x_{n+1}$ tends to zero (with respect to the p-adic absolute value) as n tends to infinity. Prove that $\{x_n\}$ is a Cauchy sequence.

Suppose $x_n - x_{n+1}$ tends to zero. Then for every ϵ, there exists an $N > 0$ such that $\left|x_n - x_{n+1}\right|_p < \epsilon$ for all $n \geq N$. Let $m > n \geq N$. Then $\left|x_k - x_{k+1}\right|_p < \epsilon$ for all $n \leq k \leq m-1$. By the strong triangle inequality,

$$\begin{aligned}\left|x_n - x_m\right|_p &= \left|(x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + \ldots + (x_{m-1} - x_m)\right|_p \\ &\leq \max\{\, \left|(x_n - x_{n+1})\right|_p, \left|(x_{n+1} - x_{n+2})\right|_p, \ldots, \left|(x_{m-1} - x_m)\right|_p \,\} < \epsilon\end{aligned}$$

The last inequality holds because each absolute value is less than ϵ. We conclude that the sequence is Cauchy.

2. Let $\{y_n\}$ be a sequence of reals in the interval $[0,1]$ such that $y_n - y_{n+1}$ tends to zero (with respect to the real absolute value) as n tends to infinity. Can we conclude that $\{y_n\}$ is a Cauchy sequence?

No. Let $h_n = \sum_{k=1}^{n} \frac{1}{k}$ be the harmonic series. This sequence is divergent by the integral test from calculus, so it cannot be Cauchy in the reals. Let $y_1 = 1$, $y_2 = 1 - \frac{1}{2}$, $y_3 = 1 - \frac{1}{2} - \frac{1}{3}$, $y_4 = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4}$, etc. In general, let $y_n = y_{n-1} \pm \frac{1}{n}$, where the $\pm$ sign is the same as the term before, unless doing so puts y_n out of the interval $[0,1]$, in which case the sign changes.

It is clear that $|y_n - y_{n+1}| = \frac{1}{n} \longrightarrow 0$ as n tends to infinity. To show y_n is not Cauchy, let $\epsilon = \frac{1}{2}$. For any $N > 0$, choose $n > N$ such that $y_n < 1 < y_n + \frac{1}{n+1}$ (this is possible since the harmonic series is unbounded). Likewise, choose m to be the smallest $m > n$ such that $y_m - \frac{1}{m+1} < 0 < y_m$. Then

$$|y_n - y_m| = \sum_{k=n+1}^{m} \frac{1}{k} > 1 - \frac{1}{n+1} - \frac{1}{m+1} > \frac{1}{2} = \epsilon$$

Hence, $\{y_n\}$ is contained in $[0,1]$ and is not Cauchy.

3. Prove that the positive integers are a dense subset of the p-adic closed ball $\overline{B(0,1)}$.

Let $x \in \overline{B(0,1)} = \mathbb{Z}_p$. Then $x = a_0 + a_1 p + a_2 p^2 + \ldots$. Let $\epsilon > 0$. It suffices to show that some postive integer is contained in $B(x,\epsilon)$. Choose n large enough such that $\frac{1}{p^{n+1}} < \epsilon$ (can choose $n = \left\lceil -\frac{\log \epsilon}{\log p} \right\rceil$). Let $y = a_0 + a_1 p + \ldots + a_n p^n$. Then

$$\left|x - y\right|_p = \left|a_{n+1}p^{n+1} + a_{n+2}p^{n+2} + \ldots\right|_p = \left|p^{n+1}\right|_p \cdot \left|a_{n+1} + a_{n+2}p + \ldots\right|_p \leq \frac{1}{p^{n+1}} < \epsilon$$

Therefore, the positive integer y lies in $B(x,\epsilon)$. We conclude that the positive integers are dense in $\overline{B(0,1)} = \mathbb{Z}_p$.

4. Suppose $x = a_0 + a_1 p + a_2 p^2 + \ldots$ is a p-adic integer. Show that the partial sums of x form a Cauchy sequence in the p-adic absolute value.

Let $x_n = a_0 + a_1 p + \ldots + a_n p^n$ be the n^{th} partial sum of $x \in \mathbb{Z}_p$. By problem #1, it suffices to show that $\left|x_n - x_{n+1}\right|_p \longrightarrow 0$:

$$\left|x_n - x_{n+1}\right|_p = \left|-a_{n+1}p^{n+1}\right|_p \leq \frac{1}{p^{n+1}}$$

(the $\leq$ is there since a_{n+1} could be zero). Since $\frac{1}{p^{n+1}}$ grows arbitrarily small, the partial sums of x form a Cauchy sequence.

5. Let $F(x)$ be a polynomial with coefficients in $\mathbb{Z}_p$ such that $F(a_0) \equiv 0 \pmod{p}$ has a solution and $F'(a_0) \not\equiv 0 \pmod{p}$. Suppose $\alpha_n = a_0 + a_1 p + \cdots + a_{n-1}p^{n-1}$ has been found so that $F(\alpha_n) \equiv 0 \pmod{p^n}$. Find $\alpha_{n+1} = \alpha_n + tp^n$ so that $F(\alpha_{n+1}) \equiv 0 \pmod{p^{n+1}}$.

First note that since $\alpha_n = a_0 + up$, by Taylor's Theorem

$$F'(\alpha_n) = F'(a_0) + F''(a_0)up + \ldots \equiv F'(a_0) \not\equiv 0 \pmod{p}$$

Now using Taylor's Theoerem on $F(\alpha_{n+1})$ and $\alpha_{n+1} = \alpha_n + tp^n$:

$$0 \equiv F(\alpha_{n+1}) \equiv F(\alpha_n) + F'(\alpha_n)tp^n \pmod{p^{n+1}}$$

Since $F(\alpha_n) \equiv 0 \pmod{p^n}$, $F(\alpha_n) = kp^n$ for some $k \in \mathbb{Z}$. Hence,

$$\begin{aligned} -kp^n = -F(\alpha_n) &\equiv F'(\alpha_n)tp^n \pmod{p^{n+1}} \\ -k &\equiv F'(\alpha_n)t \equiv F'(\alpha_0)t \pmod{p} \end{aligned}$$

Since $F'(\alpha_n) \equiv F'(a_0) \not\equiv 0 \pmod{p}$, $F'(\alpha_0)$ has a inverse modulo p. Therefore,

$$t \equiv -k\left[F'(\alpha_0)\right]^{-1} \pmod{p}$$

Since we can solve for t, we have found $\alpha_{n+1} = \alpha_n + tp^n$ which by construction satifies $F(\alpha_{n+1}) \equiv 0 \pmod{p^{n+1}}$, as desired.

6. Show that if $m \equiv 1 \pmod{8}$, then $x^2 = m$ has a solution in $\mathbb{Z}_2$. (Hint: You need to use the strong form of Hensel's Lemma.)

Let $p = 2$ and $f(x) = x^2 - m$, where m is any 2-adic integer. If $m \equiv 1 \pmod{8}$, then $f(x)$ has a root in $\mathbb{Z}_2$. To see this, let $\alpha_1 = 1$. Then $f(1) = 1 - m \equiv 0 \pmod{2^3}$ and $2^1 \,||\, f'(1) = 2$. Since $x = 3$, $y = 1$ satisfy the equation $x \geq 2y + 1$, there is a root by the Strong Version of Hensel's Lemma.