Challenge Problems 2

Math 104 - Dr. Evans
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1. Suppose that for some prime p, there exist integers a and b such that $2p = a^2 + 5b^2$. Show that there also exist integers c and d such that $3p = c^2 + 5d^2$. Just as an example, $14 = 3^2 + 5 \cdot 1^2$ and $21 = 1^2 + 5 \cdot 2^2$.

2. Prove that there are infinitely many primes of the form $4k+3$. Use a similar proof to show there are infinitely many primes of the form $6k+5$. (Hint: Mimic Euclid's proof of the infinitude of any kind of prime. See Proofs from the Book by Aigler and Ziegler for Euclid's proof.)

3. A number b is said to be a *cubic residue* modulo p if there is a solution to the equation $x^3 \equiv b \pmod{p}$. Find the cubic residues for $p = 7$, $p = 13$, and $p = 19$. How many cubic residues do you think there are for a prime p? Disprove your answer by finding the cubic residues for $p = 11$ or $p = 17$. Can you come up with the right formulation now?