HOMEWORK 6
Math 104 - Dr. Evans
UCSD Winter 2004

1. (a) Determine whether 43 is a square modulo 67.
   (b) Determine whether -35 is a square modulo 199.
   (c) Determine whether 101 is a square modulo 163.

2. (a) Prove that for *any* odd a, $\left(\frac{-1}{a}\right)$ is equal to 1 if $a \equiv 1 \pmod 4$ and equal to -1 if $a \equiv 3 \pmod 4$.
   (b) Prove that for *any* odd a, $\left(\frac{2}{a}\right)$ is equal to 1 if $a \equiv 1,7 \pmod 8$ and equal to -1 if $a \equiv 3,5 \pmod 8$.

3. In this problem, we find the general solution to the *second order recurrence relation* $q_{n+2} = Aq_{n+1} + Bq_n$ when $r^2 - Ar - B = 0$ has distinct roots.
   (a) If r_1 and r_2 are distinct roots of the *characteristic polynomial* $r^2 - Ar - B = 0$, show that $q_n = c_1 r_1^n + c_2 r_2^n$ satisfies $q_{n+2} = Aq_{n+1} + Bq_n$, where c_1 and c_2 are arbitrary constants.
   (b) The constants c_1 and c_2 are determined by the initial conditions on q_0 and q_1. Solve for c_1 and c_2 in terms of q_0 and q_1.
   (c) Find a formula for the n^{th} Fibonacci number F_n, where $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$.
   (d) Find a formula for the n^{th} Lucas number L_n, where $L_0 = 2$, $L_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$.

4. (a) Using the formulas from problem 3c and 3d, prove that $F_{2n} = F_n L_n$.
   (b) Prove that
   $$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}.$$
   (c) Prove that
   $$\lim_{n\to\infty} \frac{L_n}{F_n} = \sqrt{5}.$$

CHALLENGE PROBLEMS

A. Prove that
$$\sum_{n=1}^{\infty} F_n x^n = \frac{x}{1-x-x^2}.$$
What is the radius of convergence? Then show that
$$\frac{1}{89} = \sum_{n=1}^{\infty} \frac{F_n}{10^{n+1}} = .01 + .001 + .0002 + .00003 + .000005 + \ldots$$

B. Prove that $F_n | F_{kn}$ for any $k \geq 1$. (Hint: First prove $F_{a+b} = F_a F_{b-1} + F_{a-1} F_b$.) In particular, we get the cool result: If $n|m$, then $F_n | F_m$. For example, every third Fibonacci number is divisible by $F_3 = 2$ and every fifth Fibonacci number is divisible by $F_5 = 5$.