HINTS FOR EXTRA PROBLEMS - MIDTERM 2
Math 109 - Dr. Chow
UCSD Winter 2003

Chapter 3
26. Prove that for each natural number $n \geq 5$, $4^n > n^4$.

The trick is to expand $(n+1)^4 = n^4 + 4n^3 + 6n^2 + 4n + 1$ and expand $4^(n+1) = 4 \cdot 4^n = 4^n + 4^n + 4^n + 4^n$. By inductive hypothesis, $4^n > n^4$. Since $n \geq 5$, then $4^n > n^4 = n \cdot n^3 > 4n^3$ since $n > 4$. Likewise, $4^n > n^4 = n^2 \cdot n^2 > 6n^2$ since $n^2 > 6$. Finally, $4^n > n^4 = n^3 \cdot n > 5n > 4n + 1$. In the end, we have $4^(n+1) = 4^n + 4^n + 4^n + 4^n > (n^4) + (4n^3) + (6n^2) + (4n+1) = (n+1)^4$.

Chapter 3
65a. For each natural number n, prove that f_{3n} is divisible by 3.

$f_{3(n+1)} = f_{3n+3} = f_{3n+2} + f_{3n+1} = (f_{3n+1} + f_{3n}) + f_{3n+1} = 2 \cdot f_{3n+1} + f_{3n}$. By inductive hypothesis, f_{3n} is even, and clearly $2 \cdot f_{3n+1}$ is even, so the sum must also be even.

Chapter 4
79. Illustrate the most important aspect of exponentiation of congruence classes as defined in Exercise 78.

The trick is that if $a \equiv b \pmod{n}$, then $a^m \equiv b^m \pmod{n}$. Example:

$$4 \equiv 4 \pmod{5}$$
$$4^2 \equiv 1 \pmod{5}$$
$$4^4 = (4^2)^2 \equiv 1^2 \equiv 1 \pmod{5}$$
$$4^8 = (4^4)^2 \equiv 1^2 \equiv 1 \pmod{5}$$

Then to find $[4]^{[9]} = [4^9]$, we just multiply the congruence classes of 4 and 4^8 modulo 5. $[4]^{[9]} = [4^9] = [4^8 \cdot 4] = [4^8] \cdot [4] = [1] \cdot [4] = [4]$.

Chapter 5
23. and 24.

For 23, see this week's homework solution. For 24, try to modify the proof from 23 to work for one-to-one functions. It should be pretty similar.