Homework 1
Math 109 - UCSD Winter 2003

2. Use the definition of $\rightarrow$ to determine the conditions under which each of the following compound propositions is true.

a) If $8 > 5$*, then* $3 < 1$.

The statement $8 > 5$ is true, while the statement $3 < 1$ is false. Thus, no condition is possible which makes the proposition true.

b) If $a = 2$*, then* $1 < 3$.

The second statement $1 < 3$ is true, so the proposition is true no matter whether the first statement $a = 2$ is true or not. So no condition is necessary to make the proposition true.

c) If $5 > 8$*, then* $3 < 1$.

The first statement $5 > 8$ is false, so the proposition is true no matter whether the second statement $3 < 1$ is true or not. Hence, no condition is necessary to make the proposition true.

d) If $1 < 3$*, then* $a = 2$.

The first statement $1 < 3$ is true, while the second statement $a = 2$ is only true for one particular value of a, namely 2. Hence, we need the condition that a must be 2 to make the proposition true.

e) If $1 > 3$*, then* $a = 2$.

This proposition is always true for the same exact reason as part *c)*.

15. Prepare a truth table for each of the following expressions.

a) $P \wedge Q \rightarrow P \vee Q$

P	Q	$P \wedge Q$	$P \vee Q$	$P \wedge Q \rightarrow P \vee Q$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

b) $P \rightarrow (Q \rightarrow R)$

P	Q	R	$Q \rightarrow R$	$P \rightarrow (Q \rightarrow R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	T

25. Write, in symbols, the converse, the contrapositive, and the negation of each of the following propositional expressions.

a) $P \rightarrow (Q \vee R)$

Converse: $(Q \vee R) \rightarrow P$
Contrapositive: $\neg(Q \vee R) \rightarrow \neg P$ or $(\neg Q \wedge \neg R) \rightarrow \neg P$ (either is acceptable)

Negation: $\neg(P \to (Q \vee R)) \leftrightarrow P \wedge \neg(Q \vee R)$ which can be written also as $P \wedge \neg Q \wedge \neg R$. (This is the best possible answer.)

b) $P \to (Q \wedge R)$

Converse: $(Q \wedge R) \to P$
Contrapositive: $\neg(Q \wedge R) \to \neg P$ or $(\neg Q \vee \neg R) \to \neg P$
Negation: As before, $P \wedge (\neg Q \vee \neg R)$.

47. A sequence x_n is a **Cauchy sequence** *provided that for each $\epsilon > 0$, there is a natural number N such that if $m, n > N$, then $|x_n - x_m| < \epsilon$. Without using any negative words, state what it means to say that x_n is not a Cauchy sequence.*

Before we start, let me comment that there is a point of possible confusion in the definition above. The phrase "if $m, n > N$, then ..." is not a proposition of the form $A \to B$. It is really a $(\forall m > N)(\forall n > N)P(m,n)$ statement. So when negating the statement, you shouldn't have $m > N$ OR $n > N$ in your negation.

So the first thing we need to do is translate the definition of a Cauchy sequence into pure symbols. "For each $\epsilon > 0$" becomes $(\forall \epsilon > 0)$; "there is a natural number N" becomes $(\exists N \in \mathbb{N})$; "if $m, n > N$" becomes $(\forall m > N)(\forall n > N)$; and finally "$|x_n - x_m| < \epsilon$" is the proposition $P(m,n)$. So the definition of a Cauchy sequence becomes

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall m > N)(\forall n > N)P(m,n)$$

Negating the statement changes all $\forall$ into $\exists$ and all $\exists$ into $\forall$. Hence, a sequence is not a Cauchy sequence if

$$(\exists \epsilon > 0)(\forall N \in \mathbb{N})(\exists m > N)(\exists n > N)\neg P(m,n)$$

Translating this back into words, the negation is "There is a $\epsilon > 0$ such that for all $N \in \mathbb{N}$, there exists an $m > N$ and $n > N$ such that $|x_n - x_m| \geq \epsilon$". We'll discuss Cauchy sequences more in the last week of class.

55. Prove that $\neg[(\exists x)P(x)]$ is equivalent to $(\forall x)\neg[P(x)]$

The proof of this exactly like the proof of part *a)* of Theorem 1.5 with a few minor modifications. I suggest that you compare the proof below to the one in the text.

Suppose the proposition $\neg[(\exists x)P(x)]$ is true. Then $(\exists x)P(x)$ is false, so the truth set of $P(x)$ (the set of x that make $P(x)$ true) is empty. Therefore, the truth set of $\neg P(x)$ is the whole universe, and the proposition $(\forall x)\neg[P(x)]$ is true.

Now suppose the proposition $(\forall x)\neg[P(x)]$ is true. Then the truth set of $\neg P(x)$ is the whole universe, so the truth set of $P(x)$ is empty. Therefore, the proposition $(\exists x)P(x)$ is false, so $\neg[(\exists x)P(x)]$ is true.