

HOMEWORK 3
Math 109 - Dr. Chow
UCSD Winter 2003

18. We give three “proofs” of the following conjecture: If A and B are sets and $A \subseteq B$, then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Which, if any, are correct? Justify your answer.

All three proofs are incorrect. The justifications are given below.

“Proof 1”: Let $x \in \mathcal{P}(A)$. Then $x \in A$. Since $A \subseteq B$, $x \in B$. Therefore $x \in \mathcal{P}(B)$, so $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

By definition of the power set, $\mathcal{P}(A)$ is the set of all subsets of A . Hence, an element x of $\mathcal{P}(A)$ is a subset of A , not an element of A . So going from the statement $x \in \mathcal{P}(A)$ to $x \in A$ is a faulty step. The phrase $x \in A$ should be replaced with $x \subseteq A$. Likewise, we should replace $x \in B$ with $x \subseteq B$. So the problem in this proof is in using the definition of power set.

“Proof 2”: Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$. Then $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ and $\mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. Therefore, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

This is an example of the conjecture, not a proof. A correct proof would have to start with assuming that A and B are arbitrary sets.

“Proof 3”: Let $x \in A$. Since $A \subseteq B$, $x \in B$. Since $x \in A$ and $x \in B$, $\{x\} \in \mathcal{P}(A)$ and $\{x\} \in \mathcal{P}(B)$. Therefore, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

This one is a bit tricky. Every step along the way is true. However, one cannot conclude that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ from the previous steps. The proof shows that every one element set $\{x\}$ which is in $\mathcal{P}(A)$ is also in $\mathcal{P}(B)$. But this does not show that an **arbitrary** element of $\mathcal{P}(A)$ is also an element of $\mathcal{P}(B)$. For that, you need to show **any** subset of A is also a subset of B . This is what we’ll do in the next problem.

19. Modify an incorrect proof in Exercise 18 to obtain a correct proof of the previous conjecture.

“Proof 2” is an example and cannot be changed into a correct proof. “Proof 3” cannot be altered to show that any subset of A is a subset of B . So we have to change “Proof 1” into a correct proof. This is actually very simple, and only involves changing two \in symbols into \subseteq in the original “proof.”

Let $x \in \mathcal{P}(A)$. Then $x \subseteq A$. Since $A \subseteq B$, $x \subseteq B$ by transitivity of subsets, Proposition 2.4. Therefore $x \in \mathcal{P}(B)$, so $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. \square

33. Prove Theorem 2.6(c) and (d).

I will only prove Theorem 2.6(c), which is $(A \cap B)' = A' \cup B'$. We first show $(A \cap B)' \subseteq A' \cup B'$, then $A' \cup B' \subseteq (A \cap B)'$.

Let $x \in (A \cap B)'$. Then $x \notin A \cap B$. In words, this says that x is not in both A and B . In symbols, this means $\neg((x \in A) \wedge (x \in B))$ is true. By de Morgan’s laws (see p.12), this is equivalent to $\neg(x \in A) \vee \neg(x \in B)$ being true. Translating back into set notation, this means that $x \notin A$ or $x \notin B$, which is the same as $x \in A'$ or $x \in B'$. Hence, we have $x \in A' \cup B'$. This shows $(A \cap B)' \subseteq A' \cup B'$.

Now let $x \in A' \cup B'$. Then either $x \in A'$ or $x \in B'$, which is the same as saying $x \notin A$ or $x \notin B$. Again by de Morgan’s laws, this is equivalent to $x \in A$ and $x \in B$ being false, i.e., $x \notin A \cap B$. Hence, we have $x \in (A \cap B)'$. This shows that $A' \cup B' \subseteq (A \cap B)'$. Since each set is contained in the other, we conclude that $(A \cap B)' = A' \cup B'$. \square

35. Prove for any sets A and B , the following are true.

a) $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$

Here is a proof using the properties from Proposition 2.5, Theorem 2.6, and Theorem 2.7.

$$\begin{aligned}
 (A \cup B) - (A \cap B) &= (A \cup B) \cap (A \cap B)' && \text{By Theorem 2.6(d)} \\
 &= (A \cup B) \cap (A' \cup B') && \text{By Theorem 2.6(c)} \\
 &= ((A \cup B) \cap A') \cup ((A \cup B) \cap B') && \text{By Theorem 2.7} \\
 &= (A' \cap (A \cup B)) \cup (B' \cap (A \cup B)) && \text{By Proposition 2.5(d)} \\
 &= ((A' \cap A) \cup (A' \cap B)) \cup ((B' \cap A) \cup (B' \cap B)) && \text{By Theorem 2.7} \\
 &= (\emptyset \cup (A' \cap B)) \cup ((B' \cap A) \cup \emptyset) && \text{Because } C \cap C' = \emptyset \\
 &= (A' \cap B) \cup (B' \cap A) && \text{By Proposition 2.5(a)} \\
 &= (B \cap A') \cup (A \cap B') && \text{By Proposition 2.5(d)} \\
 &= (B - A) \cup (A - B) && \text{By Theorem 2.6(d)} \\
 &= (A - B) \cup (B - A) && \text{By Proposition 2.5(d)}
 \end{aligned}$$

This constitutes a correct proof, since I started at one side of the equation, $(A \cup B) - (A \cap B)$, and through a series of steps justified by proven theorems, I arrive at the other side of the equation, $(A - B) \cup (B - A)$. \square

This is not the only way. One can also show that every element in $(A \cup B) - (A \cap B)$ is also an element of $(A - B) \cup (B - A)$, and vice versa. But this is very similar in nature to the proof in Exercise 33.

b) $A \cap B$ and $A - B$ are disjoint.

To show two sets are disjoint, we need to show that the intersection is empty ($(A \cap B) \cap (A - B) = \emptyset$), which means that there is no element in both $A \cap B$ and $A - B$. We prove this by contradiction.

So suppose that $(A \cap B) \cap (A - B)$ is not empty. Then there exists an element $x \in (A \cap B) \cap (A - B)$. By definition of intersection, $x \in A \cap B$ and $x \in A - B$. The first statement says $x \in A$ and $x \in B$, while the second statement says that $x \in A$ and $x \notin B$. But an element cannot both be in B and not in B at the same time. This is a contradiction. Hence, our original assumption must be wrong. We conclude that $(A \cap B) \cap (A - B)$ is empty. \square