HOMEWORK 4
Math 109 - Dr. Chow
UCSD Winter 2003

1j. $7^n - 2^n$ is divisible by 5.

Let $S = \{n \in \mathbb{N} : 5 \text{ divides } 7^n - 2^n\}$. First, $1 \in S$ since $7^1 - 2^1 = 7 - 2 = 5$ is obviously divisible by 5. Now suppose that $n \in S$. This means that we are assuming $7^n - 2^n$ is divisible by 5, which by definition means that we can write $7^n - 2^n = 5k$ for some integer k. We need to show that $n + 1 \in S$.

$$\begin{aligned} 7^{n+1} - 2^{n+1} &= 7 \cdot 7^n - 2 \cdot 2^n \\ &= (5+2) \cdot 7^n - 2 \cdot 2^n \\ &= (5 \cdot 7^n + 2 \cdot 7^n) - 2 \cdot 2^n \\ &= 5 \cdot 7^n + 2 \cdot (7^n - 2^n) \\ &= 5 \cdot 7^n + 2 \cdot 5k \\ &= 5(7^n + 2k) \end{aligned}$$

This shows that $7^{n+1} - 2^{n+1}$ is divisible by 5, and hence $n+1 \in S$. By the Principal of Mathematical Induction, we conclude that $S = \mathbb{N}$, which means that $7^n - 2^n$ is divisible by 5 for all natural numbers. $\square$

37a. Prove that for each natural number n, any set with n members has $\frac{n(n-1)}{2}$ 2-element subsets.

This was proven in section. Also, the proof is very similar to part (b), which I will prove below.

37b. Use the result of (a) to show that for each natural number n, any set with n members has $\frac{n(n-1)(n-2)}{6}$ 3-element subsets.

If $n = 1$ or $n = 2$, then there are no 3-element subsets. So the base case should start with $n = 3$, which has exactly one subset, which agrees with the formula $\frac{3(3-1)(3-2)}{6} = 1$.

Now suppose any set with n members has $\frac{n(n-1)(n-2)}{6}$ 3-element subsets. Let $A = \{a_1, a_2, \ldots, a_n, a_{n+1}\}$. There are two types of 3-element subsets, those that contain the element a_{n+1} and those that don't.

Case 1: a_{n+1} is an element of the 3-element subset. In this case, the other two elements must be in $\{a_1, a_2, \ldots, a_n\}$. From part (a), there are exactly $\frac{n(n-1)}{2}$ 2-element subsets in $\{a_1, a_2, \ldots, a_n\}$.

Case 2: a_{n+1} is not an element of the 3-element subset. Then the 3-element subset must be contained in $\{a_1, a_2, \ldots, a_n\}$. By our inductive hypothesis, there are $\frac{n(n-1)(n-2)}{6}$ 3-element subsets of $\{a_1, a_2, \ldots, a_n\}$.

Thus, there are a total of $\frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{6} = \frac{3n^2-3n}{6} + \frac{n^3-3n^2+2n}{6} = \frac{n^3-n}{6} = \frac{(n+1)n(n-1)}{6}$ 3-element subsets of $\{a_1, a_2, \ldots, a_n, a_{n+1}\}$. By induction, we have that the number of 3-elements subsets of an set with n memebers is $\frac{n(n-1)(n-2)}{6}$ for all $n \geq 3$. $\square$.

66. Let $f_1, f_2, f_3, \ldots$ be the Fibonacci numbers. Prove by induction that for each natural number n, f_{5n} is a multiple of 5.

Let $S = \{n \in \mathbb{N} : 5 \text{ divides } f_{5n}\}$. By definition, $f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3, f_5 = 5$. This shows that $1 \in S$. Now suppose that $n \in S$, that is, f_{5n} is a multiple of 5. To show that $n + 1 \in S$, we want to write $f_{5(n+1)} = f_{5n+5}$ in terms of f_{5n}, which we are assuming is a multiple of 5. To do this, we use the definition $f_{k+2} = f_{k+1} + f_k$ several times:

$$\begin{aligned} f_{5n+5} = & \qquad f_{5n+4} + f_{5n+3} \\ &= (f_{5n+3} + f_{5n+2}) + f_{5n+3} = 2 \cdot f_{5n+3} + f_{5n+2} \\ &= 2 \cdot (f_{5n+2} + f_{5n+1}) + f_{5n+2} = 3 \cdot f_{5n+2} + 2 \cdot f_{5n+1} \\ &= 3 \cdot (f_{5n+1} + f_{5n}) + 2 \cdot f_{5n+2} = 5 \cdot f_{5n+1} + 3 \cdot f_{5n} \end{aligned}$$

Since $5 \cdot f5n + 1$ is a multiple of 5, and $3 \cdot f_{5n}$ is a multiple of 5 by our inductive hypothesis, the sum is also a multiple of 5. Hence, we have shown that f_{5n+5} is a multiple of 5, and so $n + 1 \in S$. We conclude by induction that f_{5n} is a multiple of 5 for all natural numbers n.