Homework 5
Math 109 - Dr. Chow
UCSD Winter 2003

36. For any two points (a, b) and (c, d) of the plane, define $(a, b) \simeq (c, d)$ provided that $a^2 + b^2 = c^2 + d^2$.

a) Prove that $\simeq$ is an equivalence relation on $\mathbb{R} \times \mathbb{R}$.

Reflexive: Since $a^2 + b^2 = a^2 + b^2$, we have $(a, b) \simeq (a, b)$.

Symmetric: Suppose that $(a, b) \simeq (c, d)$, then $a^2 + b^2 = c^2 + d^2$. This is the same as $c^2 + d^2 = a^2 + b^2$, which implies that $(c, d) \simeq (a, b)$.

Transitive: Suppose $(a, b) \simeq (c, d)$ and $(c, d) \simeq (e, f)$. Then $a^2+b^2 = c^2+d^2$ and $c^2+d^2 = e^2+f^2$ by the definition of $\simeq$. Since $a^2 + b^2$ and $e^2 + f^2$ are both equal to $c^2 + d^2$, we know that the two must be equal themselves. So $a^2 + b^2 = e^2 + f^2$ implies that $(a, b) \simeq (e, f)$. $\square$

b) List all the members of $[(0, 0)]$.

$[(0, 0)]$ is the set of all elements which are equivalent to the point $(0, 0)$. If $(a, b) \simeq (0, 0)$, then $a^2 + b^2 = 0^2 + 0^2 = 0$. This can only happen if both a and b are zero. Hence, $(a, b) \simeq (0, 0)$ if and only if $(a, b) = (0, 0)$. We conclude that $[(0, 0)] = \{(0, 0)\}$.

c) Give a geometric description of $[(5, 11)]$.

Recall that the distance formula on $\mathbb{R} \times \mathbb{R}$ is given by $|(x, y)| = \sqrt{x^2 + y^2}$. Squaring both sides, we have $|(x, y)|^2 = x^2 + y^2$. So $[(5, 11)] = \{(a, b) : a^2 + b^2 = 5^2 + 11^2\}$ is the set of all points on the plane which have the same distance to the origin as $(5, 11)$, which is a circle of radius $\sqrt{5^2 + 11^2} = \sqrt{146}$ centered at the origin.

49. Prove Theorem 4.5: Let $\mathcal{A}$ be a partition of a nonempty set S. If x and y are members of S, define $x \simeq y$ provided that some member of $\mathcal{A}$ contains both x and y. Then $\simeq$ is an equivalence relation on S.

First note that $\mathcal{A}$ is a partition if and only if every element $x \in S$ belongs to exactly one member of $\mathcal{A}$. To prove that $\simeq$ is an equivalence relation, we show that it is reflexive, symmetric, and transitive. There's very little to show in the first two cases, but the third makes use of the previously stated property of partitions.

Reflexive: Let x be any element of S. By the definition of a partition, x belongs to some member of $\mathcal{A}$. Obviously, x and x belong to the same member of $\mathcal{A}$. By definition, this shows that $x \simeq x$ for all $x \in S$.

Symmetric: Suppose that $x \simeq y$, that is, some member of $\mathcal{A}$ contains both x and y. The same member of $\mathcal{A}$ also contains both y and x, and hence $y \simeq x$.

Transitive: Suppose that $x \simeq y$ and $y \simeq z$. Then there is a member A_1 of $\mathcal{A}$ which contains both x and y, and there is a member A_2 of $\mathcal{A}$ which contains both y and z. Since the members of $\mathcal{A}$ are disjoint, two members which are not the same have an empty intersection. But both A_1 and A_2 contain the element y. Hence, A_1 and A_2 must be the same member of $\mathcal{A}$. This member contains both x and z, and so we conclude that $x \simeq z$. $\square$

68. Let $n \in \mathbb{N}$, and let $a, b \in \mathbb{Z}$. Prove that a is congruent to b, mod n, if and only if a and b have the same remainder when divided by n.

We need to use the Euclidean algorithm in both cases. The Euclidean algorithm tells us that there exist integers q_1, q_2, r_1, r_2 with $0 \leq r_1, r_2 < n$ such that $a = q_1 n + r_1$ and $b = q_2 n + r_2$. The numbers r_1 and r_2 are called the remainders when a and b are divided by n, respectively.

Assume that a is congruent to b mod n. Then $a - b = kn$ for some integer k. Using the equations above, we also have $a - b = (q_1 n + r_1) - (q_2 n + r_2) = (q_1 - q_2)n + (r_1 - r_2)$. Isolating the difference of the remainders, we have $r_1 - r_2 = (a - b) - (q_1 - q_2)n = kn - (q_1 - q_2)n$. Hence, $r_1 - r_2$ is divisible by n. Also note that since both r_1 and r_2 are between 0 and $n - 1$, we have $-(n-1) \leq r_1 - r_2 \leq (n-1)$. Since $r_1 - r_2$ is a number between $-(n-1)$ and $n - 1$ which is divisible by n, it must be equal to 0. This shows that $r_1 = r_2$, which means the remainders of a and b when divided by n are the same.

Now assume that a and b have the same remainder when divided by n. Using the same equations for a and b as before, $r_1 = r_2$. Then $a - b = (q_1 n + r_1) - (q_2 n + r_2) = (q_1 - q_2)n + (r_1 - r_2) = (q_1 - q_2)n$. Hence, there is an integer k, namely $k = q_1 - q_2$, such that $a - b = kn$. We conclude that a and b are congruent mod n. $\square$