HOMEWORK 7
Math 109 - Dr. Chow
UCSD Winter 2003

5. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$. Prove that if f is strictly decreasing, then f is one-to-one and f^{-1} is strictly decreasing.

Assume that f is strictly decreasing. By definition of strictly decreasing, $f(a) > f(b)$ whenever $a < b$. Suppose $x_1 \neq x_2$. Then either $x_1 < x_2$, in which case $f(x_1) > f(x_2)$, or $x_2 < x_1$, in which case $f(x_2) > f(x_1)$. In either case, $f(x_1) \neq f(x_2)$. This proves that $f(x)$ is one-to-one.

To show f^{-1} is strictly decreasing, we prove by contradiction. Suppose that c and d be real numbers such that $c < d$ and $f^{-1}(c) \leq f^{-1}(d)$. If $f^{-1}(c) = f^{-1}(d)$, then $c = d$, which contradicts $c < d$. If $f^{-1}(c) < f^{-1}(d)$, then since f is strictly decreasing, $c = f(f^{-1}(c)) > f(f^{-1}(d)) = d$ which is also a contradiction. Therefore, we must have $f^{-1}(c) > f^{-1}(d)$, and so f^{-1} is a strictly decreasing function.

23. Let $g : A \longrightarrow B$ and $f : B \longrightarrow C$ be functions. Prove the following result: If $f \circ g : A \longrightarrow C$ maps A onto C, then $f : B \longrightarrow C$ maps B onto C. Show by example that the converse does not hold. How does this exercise relate to Theorem 5.7(a)?

Suppose $f \circ g$ maps A onto C. Then for every $c \in C$, there exists an $a \in A$ such that $f \circ g(a) = c$. Let $b = g(a) \in B$. Then $c = f \circ g(a) = f(g(a)) = f(b)$. Hence, for every $c \in C$, there exists a $b \in B$ such that $f(b) = c$. This shows that $f : B \longrightarrow C$ maps B onto C.

The converse is "If f maps B onto C, then $f \circ g$ maps A onto B. There are many counterexamples to this fact. Let $A = B = C = \mathbb{R}$, $f(x) = x + 1$, and $g(x) = x^2$. Then f maps $\mathbb{R}$ onto $\mathbb{R}$, but $f \circ g(x) = f(g(x)) = f(x^2) = x^2 + 1$, which is not onto $\mathbb{R}$.

This exercise relates to Theorem 5.7(a) because it shows that the condition that both f and g are onto is necessary to conclude that $f \circ g$ is onto. If one function is onto but the other is not, then $f \circ g$ will not be onto. The first statement says that $f \circ g$ is onto, then f must also be onto. Does it also mean that g must be onto? No! (Can you find a counterexample?)

56a. Prove that $(f \vee g)(x) = (f(x) + g(x))/2 + |f(x) - g(x)|/2$.

By definition, the "join" of two functions f and g is

$$(f \vee g)(x) = \max\{f(x), g(x)\} = \begin{cases} f(x) & \text{if } f(x) \geq g(x) \\ g(x) & \text{if } g(x) > f(x) \end{cases}$$

Call the right side of the original equation $h(x) = \frac{f(x)+g(x)}{2} + \frac{|f(x)-g(x)|}{2}$. If $f(x) \geq g(x)$, then $f(x) - g(x) \geq 0$, so $\frac{|f(x)-g(x)|}{2} = \frac{f(x)-g(x)}{2}$. Hence, $h(x) = \frac{f(x)+g(x)}{2} + \frac{f(x)-g(x)}{2} = f(x)$.

Now suppose $g(x) > f(x)$. Then $g(x) - f(x) > 0$, so $\frac{|f(x)-g(x)|}{2} = \frac{g(x)-f(x)}{2}$. Hence, $h(x) = \frac{f(x)+g(x)}{2} + \frac{g(x)-f(x)}{2} = g(x)$. Since $h(x)$ has exactly the same values as $(f \vee g)(x)$, the two functions are equal.

56b. Find an explicit expression for $(f \wedge g)(x)$ similar to the one give above for $(f \vee g)(x)$.

$\frac{f(x)+g(x)}{2}$ represents the average of the functions $f(x)$ and $g(x)$, while $\frac{|f(x)-g(x)|}{2}$ represent half the distance between the two functions. If adding half the distance to the average gives the maximum of the two functions, then subtracting half the distance from the average should give the minimum. Hence, the equation for $(f \wedge g)(x)$ should be

$$(f \wedge g)(x) = \frac{f(x) + g(x)}{2} - \frac{|f(x) - g(x)|}{2}$$