Homework 9
Math 109 - Dr. Chow
UCSD Winter 2003

24. Complete the following table so that $\{e, a, b\}$ with the operation $*$ is a group.

*	e	a	b
e	e	a	b
a	a		
b	b		

Each row and each column needs to get exactly one of each element $\{e, a, b\}$. Every element also has to have an inverse. The table below satifies both properties.

*	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

25. Is there more than one correct answer to Exercise 24? Prove your answer.

To prove this, we break into cases based on the choices for $a * a$. We know that $e * a = a * e = a$, so $a * a \neq a$. Hence, $a * a$ must be equal to b or e.

Case 1: Suppose $a * a = e$. Since every column has to contain each element, we must have $a * b = b$. This situation looks like this:

*	e	a	b
e	e	a	b
a	a	e	
b	b	b	

But we also have $e * b = b$, so $a * b = e * b$, and multiplying by the inverse of b on both sides, we have $a * b * b^{-1} = e * b * b^{-1}$, or $a = e$. This is a contradiction, so $a * a \neq e$.

Case 2: Suppose $a * a = b$. Then we must have $a * b = b * a = e$ since a must have an inverse, which forces $b * b = a$. Thus, there is only one correct answer to Exercise 24. $\square$

3. Prove Theorem 9.7: Every Cauchy sequence is bounded.

Suppose $<x_n>$ is a Cauchy sequence. Let $\varepsilon = 1 > 0$. By Theorem 9.5, there is a real number L and a natural number N such that if $n > N$, then $|x_n - L| < 1$. This is the same as $L - 1 < x_n < L + 1$, or $|x_n| < |L| + 1$ for all $n > N$. Hence, we have a bound for x_n for large n. We also need a bound that works for x_n when $n \leq N$. Here there are only a finite number of terms, so let $K = \max\{|x_1|, |x_2|, \ldots, |x_N|\}$. Then for all $n \leq N$, $|x_n| \leq K$. Finally, we need a real number that bound $|x_n|$ *for all* n. So let $M = \max\{K, |L| + 1\}$. Then $|x_n| \leq M$ for all n, and hence $<x_n>$ is bounded. $\square$

19. Prove Theorem 9.14: If $<x_n> \to x$ and there is a natural number N such that $x_n \geq 0$ for all $n > N$, then $x \geq 0$.

We prove the theorem by contradiction. Suppose $x_n \geq 0$ for all $n > N$ but $x < 0$. Let $\varepsilon = \frac{|x|}{2} > 0$. Since the sequence converges to x, there exists an integer K such that $|x_n - x| < \varepsilon$ for all $n > K$. Plugging in $\frac{|x|}{2}$ for ε and expanding, we have $-\frac{|x|}{2} < x_n - x < \frac{|x|}{2}$, or $x - \frac{|x|}{2} < x_n < x + \frac{|x|}{2}$ for all $n > K$. Since x is negative, $x + \frac{|x|}{2} = x - \frac{x}{2} = \frac{x}{2} < 0$. Thus, for all $n > K$, $x_n < x + \frac{|x|}{2} = \frac{x}{2} < 0$, which contradicts $x_n \geq 0$ for all $n > N$. We conclude that $x \geq 0$. $\square$.