Computational Evidence of the Stark Conjectures

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The Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

For example, $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(s)$ has a simple pole with residue 1 at $s = 1$ (that is, $\lim_{s\to 1} \frac{\zeta(s)}{s-1} = 1$).

Completed Zeta Function:

$$\Lambda(s) = \pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,\zeta(s)$$

Functional Equation:

$$\Lambda(s) = \Lambda(1-s)$$

$$\boxed{\Lambda(s) = \pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,\zeta(s)}$$

Can't plug in $s = 0$ or $s = 1$, but can find $\lim_{s\to 0} \frac{\Lambda(s)}{s}$ and $\lim_{s\to 0} \frac{\Lambda(1-s)}{s}$:

$$\lim_{s\to 0} \frac{\Lambda(s)}{s} = \pi^0 \cdot \lim_{s\to 0} \frac{\Gamma(s/2)}{s} \cdot \zeta(0)$$

$$\lim_{s\to 0} \frac{\Lambda(1-s)}{s} = \pi^{-1/2} \cdot \Gamma(\tfrac{1}{2}) \cdot \lim_{s\to 0} \frac{\zeta(1-s)}{s}$$

$$\lim_{s\to 0} \frac{\Gamma(s/2)}{s} = 2 \qquad\qquad \lim_{s\to 0} \frac{\zeta(1-s)}{s} = -1$$

Solving for $\zeta(0)$:

$$2 \cdot \zeta(0) = \pi^{-1/2} \cdot \pi^{1/2} \cdot (-1) \quad \Longrightarrow \quad \boxed{\zeta(0) = -\tfrac{1}{2}}$$

Dirichlet *L*-functions

Let $\chi : \mathbb{Z} \longrightarrow \mathbb{C}$ be multiplicative ($\chi(mn) = \chi(m)\chi(n)$) and periodic ($\exists f \in \mathbb{Z}^+$ such that $\chi(n+f) = \chi(n)$ for all n).

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

Functional Equation (Hecke): $L(s,\chi) \longleftrightarrow L(1-s,\overline{\chi})$

Example

$$\chi_5(n) = \left(\frac{n}{5}\right) = \begin{cases} 1 & \text{if } n \equiv 1, 4 \pmod 5 \\ -1 & \text{if } n \equiv 2, 3 \pmod 5 \\ 0 & \text{if } n \equiv 0 \pmod 5 \end{cases}$$

$$L(s, \chi_5) = \sum_{m=0}^{\infty} \left(\frac{1}{(5m+1)^s} - \frac{1}{(5m+2)^s} - \frac{1}{(5m+3)^s} + \frac{1}{(5m+4)^s} \right)$$

Plug in $s = 0$:

$$L(0, \chi_5) \text{ ``$=$'' } \sum_{m=0}^{\infty} (1 - 1 - 1 + 1) = 0$$

Take a derivative:

$$L'(s,\chi_5)=\sum_{m=0}^{\infty}\left(-\frac{\log(5m+1)}{(5m+1)^s}+\frac{\log(5m+2)}{(5m+2)^s}+\frac{\log(5m+3)}{(5m+3)^s}-\frac{\log(5m+4)}{(5m+4)^s}\right)$$

Plug in $s=0$:

$$L'(0,\chi_5) \text{ “ = ” }$$

$$\sum_{m=0}^{\infty}\big(-\log(5m+1)+\log(5m+2)+\log(5m+3)-\log(5m+4)\big)$$

$$=\log\left(\prod_{m=0}^{\infty}\frac{(5m+2)(5m+3)}{(5m+1)(5m+4)}\right)=\log\left(\tfrac{1+\sqrt{5}}{2}\right)$$

Dedekind Zeta Functions

$$\boxed{\zeta_K(s) = \sum \frac{1}{\mathbf{N}\mathfrak{a}^s} = \prod \left(1 - \frac{1}{\mathbf{N}\mathfrak{p}^s}\right)^{-1}}$$

where the sum is taken over all integral ideals $\mathfrak{a}$ of K, the product is taken over all prime ideals $\mathfrak{p}$ of K.

Taylor series at $s = 0$:

$$\zeta_K(s) = -\frac{h_K R_K}{w_K} s^r + \dots$$

where $r = r_1 + r_2 - 1$ is the rank of unit group, h_K is the class number, R_K is the regulator, and w_K is the number of roots of unity in K.

$$(p) = \begin{cases} \mathfrak{p} \cdot \overline{\mathfrak{p}} & \text{if } p \equiv 1, 4 \pmod 5 \quad (\mathbf{N}\mathfrak{p} = p) \\ \mathfrak{p} = (p) & \text{if } p \equiv 2, 3 \pmod 5 \quad (\mathbf{N}\mathfrak{p} = p^2) \\ \mathfrak{p}^2 = (\sqrt{5})^2 & \text{if } p = 5 \qquad\qquad\qquad (\mathbf{N}\mathfrak{p} = 5) \end{cases}$$

$$\zeta_{\mathbb{Q}(\sqrt{5})}(s) = \left(1 - \frac{1}{5^s}\right)^{-1} \prod_{p \equiv 1,4(5)} \left(1 - \frac{1}{p^s}\right)^{-2} \prod_{p \equiv 2,3(5)} \left(1 - \frac{1}{p^{2s}}\right)^{-1}$$

$$= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \prod_p \left(1 - \frac{\chi_5(p)}{p^s}\right)^{-1} = \zeta(s) \cdot L(s, \chi_5)$$

$$L(s, \chi_5) = \frac{\zeta_{\mathbb{Q}(\sqrt{5})}(s)}{\zeta(s)} = \frac{-\frac{hR}{2} \cdot s + \dots}{-\frac{1}{2} + \dots} = \log\left(\varepsilon^h\right) \cdot s + \dots$$

Suppose K/k is Galois, $G = \mathrm{Gal}(K/k)$ is abelian, and $\widehat{G}$ is the group of characters of G. The Frobenius automorphism $\sigma_{\mathfrak{p}}$ is

$$\left\{ \begin{matrix} \text{Unramified} \\ \text{Primes} \end{matrix} \right\} \to \mathrm{Gal}(K/k)$$

$$\mathfrak{p} \mapsto \sigma_{\mathfrak{p}}(\alpha) \equiv \alpha^{\mathbf{N}\mathfrak{p}} \pmod{\mathfrak{P}}$$

Let S be a finite set of primes in k containing all ramified and infinite primes. The *imprimitive L_S-function of K/k and χ* is:

$$\boxed{L_S(s,\chi) = \prod_{\mathfrak{p} \notin S} \left(1 - \frac{\chi(\sigma_{\mathfrak{p}})}{\mathbf{N}\mathfrak{p}^s} \right)^{-1}}$$

Fact: $\zeta_{K,S_K}(s) = \zeta_{k,S}(s) \cdot \prod_{\substack{\chi \in \widehat{G} \\ \chi \neq \mathbf{1}}} L_S(s,\chi)$

First Order Abelian Stark Conjecture

Conjecture (Stark, 1980)**.** *Suppose S contains at least three primes and there is a prime $\mathfrak{p} \in S$ which splits completely in K/k. Fix some prime $\mathfrak{P}$ in K lying above $\mathfrak{p}$. Then there is an $\varepsilon \in K^\times$ with the following properties:*

1. *ε is a $\mathfrak{p}$-unit, that is, $\left|\varepsilon\right|_{\mathfrak{Q}} = 1$ for all $\mathfrak{Q} \nmid \mathfrak{p}$.*
2. *For all $\chi \in \widehat{G}$,*

$$L'_S(0,\chi) = -\frac{1}{w_K} \sum_{\sigma \in G} \chi(\sigma) \log \left|\varepsilon^\sigma\right|_{\mathfrak{P}}$$

3. *$K(\varepsilon^{1/w_K})$ is an abelian extension of k.*

Example

$K = \mathbb{Q}(\sqrt{5})$, $k = \mathbb{Q}$, $G = \{1, \sigma : \sqrt{5} \mapsto -\sqrt{5}\}$, $\widehat{G} = \{\mathbf{1}, \chi_5 = \left(\frac{\cdot}{5}\right)\}$

$S = \{5, \infty\}$, $\mathfrak{p} = \infty$, Stark Unit: $\boxed{\varepsilon = \frac{5-\sqrt{5}}{2}}$

$$L_S'(0,\mathbf{1}) = -\frac{1}{2}\left(\log\left|\frac{5-\sqrt{5}}{2}\right| + \log\left|\frac{5+\sqrt{5}}{2}\right|\right) = -\frac{1}{2}\log 5$$

$$L_S'(0,\chi_5) = -\frac{1}{2}\left(\log\left|\frac{5-\sqrt{5}}{2}\right| - \log\left|\frac{5+\sqrt{5}}{2}\right|\right) = \log\left(\frac{1+\sqrt{5}}{2}\right)$$

Stark units are normalized Gauss sums when $k = \mathbb{Q}$ and elliptic units when k is imaginary quadratic.

Calculating Class Fields

- Approximate the L-functions at $s = 0$ to high enough precision.
- Recover the decimal approximation of the conjugates of ε.
- Take $f(x) = \prod_{\sigma \in G} (x - \varepsilon^{\sigma})$, coefficients in k.
- Recognize the coefficients in $\mathcal{O}_k$, get polynomial over $\mathbb{Z}$.

Has been used to compute the Hilbert class field for:

- Real quadratic fields of discriminant < 10,000
- Totally real cubic fields of discriminant < 150,000
- Totally real quartic fields of discriminant < 600,000

$$\begin{tikzcd}
K \arrow[d, dash] \\
H \arrow[d, dash] \\
k
\end{tikzcd}$$

$k = \mathbb{Q}(\alpha)$, $\alpha^3 - 22\alpha - 25 = 0.$

$H =$ Hilbert class field

$K =$ ray class field at $\infty_1 : \alpha \mapsto \alpha^{(1)} = -1.218623\ldots$

$G = \langle \sigma \rangle \cong \mathbb{Z}/4\mathbb{Z}$, $\widehat{G} = \langle \chi \rangle \cong \mathbb{Z}/4\mathbb{Z}$

$$L'(0, \chi) = -4.929324 + 3.979877i$$

$$L'(0, \chi^3) = -4.929324 - 3.979877i$$

$$L'(0, \mathbf{1}) = L'(0, \chi^2) = 0$$

$\varepsilon^{\sigma^0} = 138.286065\ldots$ $\qquad$ $\varepsilon^{\sigma^1} = 0.0186879\ldots$

$\varepsilon^{\sigma^2} = 0.00723138\ldots$ $\qquad$ $\varepsilon^{\sigma^3} = 53.510453\ldots$

Note that $\varepsilon^{\sigma^2} = \varepsilon^{-1}$, so the polynomial ε satisfies is

$$f(x) = \prod_{\sigma \in G} (x - \varepsilon^{\sigma}) = x^4 - s_1 x^3 + s_2 x^2 - s_1 x + 1$$

where $s_1 = 191.822438\ldots$ and $s_2 = 7404.721392\ldots$.

In this case, $s_1 = 224 + 13\alpha - 11\alpha^2$ and $s_2 = 8659 + 515\alpha - 422\alpha^2$.

$$\begin{aligned} F(x) &= x^{12} - 188\,x^{11} + 6676\,x^{10} + 27247\,x^9 + 32746\,x^8 - 3959\,x^7 \\ &\quad - 32630\,x^6 - 3959\,x^5 + 32746\,x^4 + 27247\,x^3 + 6676\,x^2 - 188\,x + 1 \end{aligned}$$

Then H should be generated by $\varepsilon + \varepsilon^{-1}$ which satisfies:

$$h(x) = x^6 - 188\,x^5 + 6670\,x^4 + 28187\,x^3 + 6051\,x^2 - 86640\,x - 84772$$

The Abelian Condition

For $K = \mathbb{Q}(\sqrt{5})$, $\varepsilon = \frac{5-\sqrt{5}}{2}$, $\eta = \sqrt{\varepsilon} = \sqrt{\phi^{-1}\sqrt{5}}$.

$\mathrm{Gal}(\mathbb{Q}(\eta)/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$ is abelian.

For $k = \mathbb{Q}(\alpha)$, $\alpha^3 - 22\alpha - 25 = 0$, K ray class field of ∞_1.

$\eta = \sqrt{\varepsilon}$ satisfies

$$\begin{aligned} f(x) = {} & x^{12} - 22\,x^{11} + 148\,x^{10} - 379\,x^9 + 724\,x^8 - 1039\,x^7 + 1150\,x^6 \\ & - 1039\,x^5 + 724\,x^4 - 379\,x^3 + 148\,x^2 - 22\,x + 1 \end{aligned}$$

ε is a square in K, so the abelian condition is trivial!
(Dummit & Hayes, 1996)

Generalized Stark Conjectures

Splitting prime in $S \quad \Longrightarrow \quad L_S(0,\chi) = 0$ for all $\chi \in \widehat{G}$.

What if no splitting prime and all $L_S(0,\chi)$ are zero?

Great thesis problem!

Have shown that GSC follows from SC under certain conditions.

Still haven't shown GSC is true for $k = \mathbb{Q}$.

Abelian condition does not hold in all situations!