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New Settings of the First Order Stark Conjectures

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The dissertation of Stefan A. Erickson is approved, and it is acceptable in quality and form for publication on microfilm:

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To my parents, for their unending love and support.

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VITA

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ABSTRACT OF THE DISSERTATION

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The First Order Abelian Stark Conjecture establishes a connection between analytic and algebraic number theory. In the 1970's, Harold Stark [10] conjectured the existence of certain algebraic units which evaluate the first derivatives of abelian L-functions at $s = 0$. Furthermore, certain roots of these algebraic units explicitly generate maximal abelian extensions of the base field. Hence, Stark conjecturally provides an answer to Hilbert's Twelfth Problem, which asks for a method of constructing abelian extensions of number fields using analytic functions.

The First Order Abelian Stark Conjecture requires that all the L-functions of a given extension K/k vanish at $s = 0$. This requirement has traditionally been satisfied by supposing that some prime of k splits completely in the extension K/k. However, there are other situations where all L-functions vanish at $s = 0$. The main goal of this thesis is to extend the conjecture to this more general setting.

After setting up notation and motivation for the Extended First Order Abelian Stark Question, we will state the question and reduce it to proving that the Stark units from intermediate fields are certain powers of elements in the top field. We prove that the reduction is satisfied under certain conditions. Finally we provide some explicit examples which test the boundaries of the extension.

Chapter 1

The Stark Conjectures

1.1 Introduction

Zeta functions and L-functions have been studied by number theorists for over two and half centuries. The *Riemann zeta function* is defined for $\mathrm{Re}(s) > 1$ as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Euler showed that this function has an infinite production expansion

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

He used the pole at $s = 1$ to show the sum $\sum \frac{1}{p}$ diverges, thus demonstrating that the primes make up a significant proportion of the natural numbers. The terms $\left(1 - \frac{1}{p^s}\right)^{-1}$ are called *Euler factors*.

In 1837, Dirichlet considered a variant of the Riemann zeta function. Dirichlet replaced the numerator with a function $\chi : \mathbb{N} \longrightarrow \mathbb{C}$ which was periodic (for some $m > 0$, $\chi(k+m) = \chi(k)$ for all $k \in \mathbb{N}$) and multiplicative ($\chi(kl) = \chi(k)\chi(l)$ for all $k, l \in \mathbb{N}$). Such functions χ are called *Dirichlet characters*. To the modified zeta functions, Dirichlet gave the name *L-functions*:

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

When χ is the trivial character (i.e., $\chi(n) = 1$ for all $n \in \mathbb{N}$), $L(s, \chi)$ is just the Riemann zeta function. When χ is not the trivial character, the series and product converges for $\mathrm{Re}(s) > 0$. By demonstrating that $L(1, \chi) \neq 0$, Dirichlet proved that there are infinitely many primes in any arithmetic sequence (as long as the initial term and the common difference are relatively prime).

In 1859, Riemann demonstrated that the zeta function satisfies a functional equation relating the value of $\zeta(s)$ to the value of $\zeta(1 - s)$. In this fashion, he analytically continued the zeta function to the whole complex plane except for a simple pole at $s = 1$. The functional equation guaranteed that $\zeta(s)$ would equal zero at all the negative even integers. Riemann conjectured that all the zeroes in the "critical strip" $0 \leq \mathrm{Re}(s) \leq 1$ would land on the line $\mathrm{Re}(s) = \frac{1}{2}$. This conjecture is known as the Riemann Hypothesis and is one of the Clay Institute Millennium Problems.

Dedekind generalized the Riemann zeta function for general number fields. A *number field* K is a finite extension of the rational numbers, obtained by adding the root of a polynomial which is irreducible over the rationals. There is an analogue of the integers inside a number field called the *ring of integers*, denoted as $\mathcal{O}_K$. (Note: We shall use the term *algebraic integers* to refer to elements of $\mathcal{O}_K$ and *rational integers* to refer to elements of $\mathbb{Z}$.) Unlike the rational integers, $\mathcal{O}_K$ may not have the property that every algebraic integer factors uniquely into prime integers. This causes the structure of $\mathcal{O}_K$ to be much more complicated than $\mathbb{Z}$.

Dedekind solved this dilemma by introducing the notion of *ideals*, which are additive subgroups of $\mathcal{O}_K$ closed under multiplication by elements of $\mathcal{O}_K$. In the integers, the ideals are just the set of all multiples of an integer n. For number fields, ideals can always be generated by at most two algebraic integers $\alpha_1, \alpha_2 \in \mathcal{O}_K$:

$$\mathfrak{A} = (\alpha_1, \alpha_2) = \left\{ \alpha_1\gamma_1 + \alpha_2\gamma_2 \;\middle|\; \gamma_1, \gamma_2 \in \mathcal{O}_K \right\}$$

Multiplication and divisibility of ideals are well-defined operations and have similar properties as they do for algebraic integers. *Prime ideals* are simply ideals that are divisible only by themselves and the "unit" ideal (equal to all of $\mathcal{O}_K$). Kummer showed that ideals uniquely factor into prime ideals. We will often say "prime" rather than "nonzero prime ideal."

As an example, consider the quadratic number field

$$K = \mathbb{Q}(\sqrt{-5}) = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Q}\}.$$

The ring of integers for K is

$$\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}.$$

This ring does not have unique factorization, as seen in the example $2 \cdot 3 = 6 = (1+\sqrt{-5})(1-\sqrt{-5})$. To demonstrate how ideals factor into prime ideals, consider the following factorizations:

$$\begin{aligned} (3) &= (3,\, 1+\sqrt{-5}) \cdot (3,\, 1-\sqrt{-5}) \\ (11) &= (11) \\ (5) &= (\sqrt{-5})^2 \end{aligned}$$

The power $\mathsf{e} = \mathsf{e}(\mathfrak{P}|p)$ to which a prime ideal in $\mathcal{O}_K$ appears in the prime factorization of p is called the *ramification index of* $\mathfrak{P}$. If the ramification index is greater than 1, we say the prime p *ramifies* (as in the third example). There are only a finite number of primes which ramify in a given extension. If all the exponents the prime factorization equal 1, then the prime p is said to be *unramified* (as in the first and second examples). If an unramified prime p does not factor in $\mathcal{O}_K$, we say the prime p is *inert* (as is example 2). Otherwise, the prime p *splits.* A prime integer can split into at most n prime ideals in $\mathcal{O}_K$, where n is the degree of the extension of K over $\mathbb{Q}$. If p splits into exactly n prime ideals in $\mathcal{O}_K$, then p *splits completely* (as in example 1, the degree here being $n = 2$).

Fix some prime ideal $\mathfrak{P}$ which appears in the factorization of (p) in $\mathcal{O}_K$. We say that $\alpha \equiv \beta \pmod{\mathfrak{P}}$ if $\beta - \alpha \in \mathfrak{P}$. The *norm* of a prime ideal (denoted as $\mathbf{N}\mathfrak{P}$) is the number of congruence classes of $\mathcal{O}_K$ modulo $\mathfrak{P}$. Alternatively, one may think of $\mathcal{O}_K/\mathfrak{P}$ as a finite field and $\mathbf{N}\mathfrak{P}$ as the size of $\mathcal{O}_K/\mathfrak{P}$. Since $\mathcal{O}_K/\mathfrak{P}$ is an extension of $\mathbb{Z}/p\mathbb{Z}$, $\mathbf{N}\mathfrak{P} = \#(\mathcal{O}_K/\mathfrak{P}) = p^{\mathsf{f}}$ for some integer $\mathsf{f} \geq 1$. $\mathsf{f} = \mathsf{f}(\mathfrak{P}|p)$ is called the *residual degree* of $\mathfrak{P}$ over p. We extend the norm to all ideals of $\mathcal{O}_K$ by multiplicativity:

$$\mathfrak{A} = \mathfrak{P}_1^{a_1} \cdots \mathfrak{P}_m^{a_m} \quad \Longrightarrow \quad \mathbf{N}\mathfrak{A} = \left(\mathbf{N}\mathfrak{P}_1\right)^{a_1} \cdots \left(\mathbf{N}\mathfrak{P}_m\right)^{a_m}$$

We can now define the *Dedekind zeta function*. Given a number field K, let

$$\zeta_K(s) = \sum \frac{1}{\left(\mathbf{N}\mathfrak{A}\right)^s} = \prod \left(1 - \frac{1}{\left(\mathbf{N}\mathfrak{P}\right)^s}\right)^{-1},$$

where the sum is taken over all nonzero ideals of $\mathcal{O}_K$ and the product is taken over all nonzero prime ideals of $\mathcal{O}_K$. The sum and product both converge for $\mathrm{Re}(s) > 1$ and can be analytically continued to the whole complex plane except for a simple pole at $s = 1$. Note that when $K = \mathbb{Q}$, the Dedekind zeta function is exactly the same as the Riemann zeta function. (For the functional equation of the Dedekind zeta function, see [13, pg. 35].)

The residue of $\zeta_K(s)$ at $s = 1$ contains some interesting algebraic information about the number field K:

$$\lim_{s \to 1} (s-1) \cdot \zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} hR}{W_K \sqrt{|D|}}$$

where r_1 is the number of real embeddings of K, r_2 is one half the number of complex embeddings (which come in conjugate pairs), h is the class number, R is the regulator, W_K is the number of roots of unity in K, and D is the discriminant of K. Via the functional equation, this residue gives information about the first nonzero term of the power series of $\zeta_K(s)$:

$$\zeta_K(s) = -\frac{hR}{W_K} \cdot s^r + (\text{higher order terms}) \tag{1.1}$$

where $r = r_1 + r_2 - 1$ is the rank of the unit group of $\mathcal{O}_K$. This formula is much simpler to work with. Henceforth, the focus of this thesis is on the special value of zeta functions and L-functions at $s = 0$.

There is a connection between the Dedekind zeta function and the Dirichlet L-functions. One can define multiplication of two Dirichlet characters χ_1 and χ_2

by taking the primitive character χ corresponding to $\chi(k) = \chi_1(k) \cdot \chi_2(k)$. Under this rule, the Dirichlet characters form a group with the trivial character $\chi_0(k) = 1$ as the identity and $\overline{\chi}$ is the inverse of χ.

Let $\widehat{G}$ be a finite group of primitive Dirichlet characters (the notation $\widehat{G}$ will be explained later). There is a corresponding number field K given explicitly through Galois theory. For this number field, we have the following theorem:

$$\zeta_K(s) = \prod_{\chi \in \widehat{G}} L(s, \chi) \tag{1.2}$$

By studying the special values of L-functions, one may hope to obtain arithmetic information about these extensions of the rational numbers. This theme is played over and over again in number theory, and is the main motivation of the Stark Conjectures. The goal of this thesis is to present a variation on the theme of Stark.

1.2 The First Order Abelian Stark Conjecture

As a preliminary to stating the First Order Abelian Stark Conjecture, we give one specific example. Suppose $K = \mathbb{Q}(\sqrt{D})$ is a real quadratic field of discriminant $D > 0$. K has exactly two characters associated to it, the trivial character and the Kronecker symbol $\chi(k) = \left(\frac{D}{k}\right)$. The L-function attached to χ is

$$L(s, \chi) = \prod_{p \nmid D} \left(1 - \frac{\left(\frac{D}{p}\right)}{p^s}\right)^{-1} \tag{1.3}$$

By the product formula for the Dedekind zeta function (equation 1.2),

$$\zeta_K(s) = \zeta(s) \cdot L(s, \chi).$$

Equation 1.1 shows that $\zeta(0) = -\frac{1}{2}$, $\zeta_K(0) = 0$, and $\zeta_K'(0) = -\frac{hR}{2}$. In this case, $R = \log(\varepsilon)$, where ε is the fundamental unit of K. Since $\zeta_K(0) = 0$ and $\zeta(0) \neq 0$,

$L(0, \chi) = 0$. Taking the derivative of equation 1.3 and evaluating at $s = 0$,

$$\begin{aligned} \zeta_K'(s) &= \zeta(s) \cdot L'(s, \chi) + \zeta'(s) \cdot L(s, \chi) \\ \zeta_K'(0) &= \zeta(0) \cdot L'(0, \chi) \\ L'(0, \chi) &= \frac{\zeta_K(0)}{\zeta(0)} = \frac{-(h \log(\varepsilon))/2}{-1/2} = h \log(\varepsilon) = \log(\varepsilon^h). \end{aligned}$$

Hence, the derivative of the L-function can be evaluated at $s = 0$ by an algebraic unit in K. In general, the First Order Abelian Stark Conjecture predicts the existence of an algebraic unit which gives information about the first derivative at $s = 0$ of all the L-functions simultaneously.

Note that $\varepsilon^h = \exp\big(L'(0, \chi)\big)$. By approximating the value of the L-function at $s = 0$ to several decimal places, one may find the integral polynomial which ε^h satisfies. More generally, the First Order Abelian Stark Conjecture predict that the L-functions can be used to find the polynomial that these algebraic units satisfy.

Before stating the First Order Abelian Stark Conjecture, we must define L-functions for an abelian extension of number fields. First, we extend the definition of Dirichlet characters to abelian Galois groups. An extension of number fields K/k is *Galois* if K contains all the roots of an irreducible polynomial over k whenever it contains one root. For example, $\mathbb{Q}(\sqrt{D})/\mathbb{Q}$ is Galois since the roots $\{\sqrt{D}, -\sqrt{D}\}$ of $f(x) = x^2 - D$ are both elements of $\mathbb{Q}(\sqrt{D})$. On the other hand, $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not Galois, since the other two roots of $f(x) = x^3 - 2$ besides $\sqrt[3]{2}$ are complex, which cannot be contained in $\mathbb{Q}(\sqrt[3]{2})$.

If K/k is Galois, then the automorphisms of K which pointwise fix the base field k form a finite group under composition, called the *Galois group* and denoted as $G = \text{Gal}(K/k)$. We will use σ and τ to denote elements of G. We shall denote the action of σ on an element $\alpha \in K$ as α^σ.

When the Galois group is abelian (that is, every automorphism commutes with each other), K/k is called an *abelian extension*. Throughout this thesis, K/k will always be assumed to be abelian unless explicitly stated. For simplicity, we will only assert that K/k is abelian in the statement of conjectures and theorems.

The set of homomorphisms $\chi : G \longrightarrow \mathbb{C}^\times$ forms a group of characters, denoted as $\widehat{G}$. It is a standard fact that when G is abelian, $\widehat{G}$ is isomorphic to G [13, Ch. 3].

As before, a prime $\mathfrak{p}$ of k factors uniquely into primes in K. For Galois extensions, the ramification indexes $\mathsf{e}_i = \mathsf{e}(\mathfrak{P}_i|\mathfrak{p})$ are all equal and the residual degrees $\mathsf{f}_i = \mathsf{f}(\mathfrak{P}_i|\mathfrak{p})$ are all equal. Letting g equal the number of distinct ideals in the factorization of $\mathfrak{p}$ in $\mathcal{O}_K$, we have the following formulas:

$$\mathfrak{p}\mathcal{O}_K = \mathfrak{P}_1^{\mathsf{e}} \cdots \mathfrak{P}_{\mathsf{g}}^{\mathsf{e}}$$
$$n = \mathsf{e}\,\mathsf{f}\,\mathsf{g}$$

$\mathfrak{P}$ is said to *lie above* $\mathfrak{p}$ if $\mathfrak{P}$ is one of the primes in the factorization of $\mathfrak{p}$ in $\mathcal{O}_K$.

Fix some prime ideal $\mathfrak{P}$ in K lying above $\mathfrak{p}$ in k. There is an automorphism $\sigma_\mathfrak{p} \in \mathrm{Gal}(K/k)$ called the *Frobenius automorphism*:

$$\alpha^{\sigma_\mathfrak{p}} \equiv \alpha^{\mathbf{N}\mathfrak{p}} \pmod{\mathfrak{P}} \quad \text{for all } \alpha \in \mathcal{O}_K.$$

The fact that $\sigma_\mathfrak{p}$ doesn't depend on the choice of $\mathfrak{P}$ follows from K/k being abelian. If $\mathfrak{p}$ is unramified in K/k, then $\sigma_\mathfrak{p}$ is uniquely determined (otherwise it is determined up to a coset of the inertia group, which is defined below). The order of $\sigma_\mathfrak{p}$ is the residual degree f.

Two other important concepts are the *decomposition group* (denoted as $G_\mathfrak{p}$) and the *inertia group* (denoted as $I_\mathfrak{p}$). Fix a finite prime $\mathfrak{P}$ in K lying above $\mathfrak{p}$ in k. Then

$$G_\mathfrak{p} = \left\{\sigma \in G \;\middle|\; \mathfrak{P}^\sigma = \mathfrak{P}\right\}$$
$$I_\mathfrak{p} = \left\{\sigma \in G \;\middle|\; \alpha^\sigma \equiv \alpha \pmod{\mathfrak{P}}\right\}$$

Again, the independence of choice of $\mathfrak{P}$ follows from K/k being abelian. If $\mathfrak{p}$ is unramified, then the inertia group is trivial and the decomposition group is a cyclic group of size f generated by the Frobenius automorphism $\sigma_\mathfrak{p}$. When $\mathfrak{p}$ is ramified, the decomposition group is the direct product of the inertia group and the cyclic group generated by some choice of the Frobenius element.

In particular, the size of decomposition group and the inertia group are as follows:

$$|G_{\mathfrak{p}}| = \mathsf{e}\,\mathsf{f}$$
$$|I_{\mathfrak{p}}| = \mathsf{e}$$

One may view the Galois group as acting on the primes $\mathfrak{P}$ above $\mathfrak{p}$. The decomposition group is the stabilizer of this group action. There is a natural homomorphism from $G_{\mathfrak{p}}$ onto the Galois group of the residual fields $\overline{G} = \mathrm{Gal}\big((\mathcal{O}_K/\mathfrak{P})/(\mathcal{O}_k/\mathfrak{p})\big)$, and the inertia group is the kernel of this map.

It is often convenient to consider the embeddings of a number field into the real numbers or complex numbers as "infinite primes." (as opposed to "finite primes," which denote the prime ideals of $\mathcal{O}_K$). An *embedding* φ of a number field is determined by which root of the irreducible polynomial is added to the rationals. The embedding φ corresponds to a *real infinite prime* if the root is real. If the root is complex, then the pair of conjugate embeddings φ and $\overline{\varphi}$ correspond to a *complex infinite prime*. If K/k is an extension of number fields, then an infinite prime Φ of K *divides* an infinite prime φ of k if $\Phi\big|_k = \varphi$. By convention, a real infinite prime φ of k splits completely in K if all infinite prime Φ of K which divide φ are real and ramifies if they are all complex. A complex prime φ of k always is considered to split completely in K.

If φ is a real infinite prime of k and Φ is a complex infinite prime of K which lies above φ, we say that the decomposition group is generated by the automorphism of K which acts as complex conjugation on Φ and which fixes the real infinite prime φ of k. Otherwise, the decomposition group is trivial for infinite primes.

In general, the decomposition group measures how far a prime is from splitting completely in K/k, whereas the inertia group measure how much a prime ramifies. The most important fact that we need is that a prime splits completely in K/k if and only if the decomposition group is trivial. More generally, the fixed field of the decomposition group is the maximal intermediate field of K/k in which the prime splits completely.

From now on, we will use the word “prime” to mean either a finite prime or an infinite prime. We will denote v as a generic prime of k and w as a generic prime of K. The qualifiers “finite” and “infinite” will be added when necessary.

Let α be an nonzero algebraic integer in $\mathcal{O}_k$ and $v = \mathfrak{p}$ be a finite prime of k. By unique factorization of ideals, $\mathfrak{p}^n$ exactly divides into the princial ideal (α) for some nonnegative integer n. We call this integer the *order of α with respect to $\mathfrak{p}$* and denote $n = \operatorname{ord}_{\mathfrak{p}}(\alpha)$. This definition extends to all of k via the relation $\operatorname{ord}_{\mathfrak{p}}\left(\frac{\alpha}{\beta}\right) = \operatorname{ord}_{\mathfrak{p}}(\alpha) - \operatorname{ord}\mathfrak{p}(\beta)$. Two important properties are the following:

$$\operatorname{ord}_{\mathfrak{p}}(\alpha\beta) = \operatorname{ord}_{\mathfrak{p}}(\alpha) + \operatorname{ord}_{\mathfrak{p}}(\beta)$$
$$\operatorname{ord}_{\mathfrak{p}}(\alpha+\beta) \geq \min\left\{\operatorname{ord}_{\mathfrak{p}}(\alpha), \operatorname{ord}_{\mathfrak{p}}(\beta)\right\}$$

For each finite prime $\mathfrak{p}$, define a (normalized) absolute value:

$$\left|\alpha\right|_{\mathfrak{p}} = \left(\mathbf{N}\mathfrak{p}\right)^{-\operatorname{ord}_{\mathfrak{p}}(\alpha)}$$

These absolute values satisfy similar properties as the real or complex absolute values, which follow from the above properties on the $\operatorname{ord}_{\mathfrak{p}}$ function:

$$\left|\alpha\beta\right|_{\mathfrak{p}} = \left|\alpha\right|_{\mathfrak{p}}\left|\beta\right|_{\mathfrak{p}}$$
$$\left|\alpha+\beta\right|_{\mathfrak{p}} \leq \max\left\{\left|\alpha\right|_{\mathfrak{p}}, \left|\beta\right|_{\mathfrak{p}}\right\}$$

The second condition is a stronger version of the triangle inequality and makes $\left|\cdot\right|_{\mathfrak{p}}$ into a *nonarchimedean absolute value*.

In the spirit of extending these definitions to the infinite primes, the absolute value for the real and complex embeddings are defined as follows:

$$\left|\alpha\right|_v = \begin{cases} |\alpha| & \text{if } v \text{ is real,} \\ |\alpha|^2 & \text{if } v \text{ is complex.} \end{cases}$$

Lemma 1.1. *Let K/k be an extension of number fields and let w be a prime in K lying above the prime v in k. Then for any $\alpha \in k$, $\left|\alpha\right|_v = \left|\alpha\right|_w^{1/|G_v|}$.*

Proof. If v and w are infinite, then

$$|G_v| = \begin{cases} 1 & \text{if } v \text{ and } w \text{ are both real or both complex,} \\ 2 & \text{if } v \text{ is real and } w \text{ is complex.} \end{cases}$$

The result then follows immediately from the definitions. If v and w are finite, let e be the ramification index of w over v and f be the residual degree of w over v. Note that $\mathbf{N}w = (\mathbf{N}v)^{\mathsf{f}}$ and $\operatorname{ord}_w(\alpha) = \mathsf{e}\operatorname{ord}_v(\alpha)$. Hence,

$$\left|\alpha\right|_w = (\mathbf{N}w)^{-\operatorname{ord}_w(\alpha)} = (\mathbf{N}v^{\mathsf{f}})^{-\mathsf{e}\operatorname{ord}_v(\alpha)} = \left|\alpha\right|_v^{|G_v|}. \quad \square$$

We now define imprimitive L-functions for the extension K/k. Let S be a finite set of primes in k, including all infinite and ramified primes. Let χ be a character in $\widehat{G}$. The *imprimitive L-function* is defined as

$$L_S(s,\chi) = \prod_{\mathfrak{p}\notin S}\left(1 - \frac{\chi(\sigma_{\mathfrak{p}})}{(\mathbf{N}\mathfrak{p})^s}\right)^{-1}.$$

As before, the product converges on $\operatorname{Re}(s) > 1$ and can be analytically continued to the whole complex plane (except for a simple pole at $s = 1$ when χ is the trivial character).

One important fact about imprimitive L-functions is the order of vanishing at $s = 0$.

Lemma 1.2. *The order of vanishing of $L_S(s,\chi)$ at $s = 0$ is equal to $|S| - 1$ if χ is the trivial character, and is equal to the number of primes v in S for which χ is trivial on G_v.*

Proof. See [12, p. 24–25] or [2, p. 12–13]. $\square$

We are now ready to state the First Order Abelian Stark Conjecture.

Conjecture 1.3 ($\operatorname{St}(K/k,S)$)**.** *Let K/k be an abelian extension of number fields. Let G be the Galois group of K/k and $\widehat{G}$ be the group of characters on G. Let S be a set of primes in k containing all infinite and ramified primes. Suppose that S contains at least two primes, including a distinguished prime v_0 which splits* *completely in K. Fix some w_0 lying above v_0. Then there exists an $\varepsilon \in K^\times$, unique up to root of unity, with the following properties:*

(i) $\left|\varepsilon\right|_w = 1$ *for all primes w not lying above a prime in S, i.e., ε is an S-unit. If $S = \{v_0, v'\}$, then for a fixed w' lying above v', $\left|\varepsilon^\sigma\right|_{w'} = \left|\varepsilon\right|_{w'}$ for all $\sigma \in G$. If $|S| \geq 3$, then $\left|\varepsilon\right|_w = 1$ for all w not lying above v_0.*

(ii) *For all $\chi \in \widehat{G}$,*

$$L'_S(0,\chi) = -\frac{1}{W_K}\sum_{\sigma\in G}\chi(\sigma)\log\left|\varepsilon^\sigma\right|_{w_0}.$$

(iii) $K(\varepsilon^{1/W_K})$ *is an abelian extension of k.*

Some comments are necessary to emphasize the important aspects of these conjectures. First, the prime factorization and absolute values of ε (called the *Stark unit*) are completely determined by the first two properties. Furthermore, the Stark unit and its conjugates evaluate all the first derivatives of the L-functions at $s = 0$ simultaneously. This relationship between the L-functions and the Stark units gives an explicit connection between analytic functions and algebraic numbers.

There is an alternate formulation of the First Order Abelian Stark Conjecture when the prime v_0 is finite, called the Brumer-Stark Conjecture. We define the *Stickelberger element* $\theta_S(0)$ in the group ring $\mathbb{Z}[G]$. Let

$$\omega_S(s) = \sum_{\chi\in\widehat{G}} L_S(s,\chi)e_\chi \in \mathbb{C}[G]$$

where $e_\chi = \frac{1}{|G|}\sum\chi(\sigma)\sigma^{-1}$. A theorem due to Deligne and Ribet shows that $\theta_S(0) := W_K\omega_S(0) \in \mathbb{Z}[G]$. The Brumer-Stark Conjecture claims that $\theta_S(0)$ is an annihilator of the class group, that is, the Stickelberger element applied to any prime $\mathfrak{P}$ in K is generated by a Stark unit:

$$\mathfrak{P}^{\theta_S(0)} = \left(\varepsilon_{\mathfrak{P}}\right)$$

Furthermore, the ${W_K}^{\text{th}}$ root of $\varepsilon_{\mathfrak{P}}$ produces an abelian extension of k when adjoined to K. It can be shown that the Brumer-Stark Conjecture for K/k and S is equivalent to the First Order Abelian Stark Conjecture for K/k and $S \cup \{\mathfrak{p}\}$ for all $\mathfrak{p} \notin S$ which split completely in K/k [12, IV.6.5].

The third property is even more intriguing. The fact that $K(\varepsilon^{1/W_K})$ is abelian over K is rather trivial; the claim that it is abelian over the base field k is quite remarkable. Generating and classifying abelian extensions of number fields is the purview of class field theory. Unfortunately, class field theory usually predicts the existence of ray class fields without giving an explicit construction. Hilbert's Twelfth Problem asks to find analytic functions which produce the explicit construction of class fields. Hilbert had in mind the two cases known at the time, the exponential functions for the rationals and elliptic functions for imaginary quadratic fields.

The First Order Abelian Stark Conjecture claims that L-functions can be used for generating abelian extensions over totally real number fields and over number fields with precisely one complex prime. The conjecture was first checked numerically by Stark [8, 9, 10] and extended by Dummit, Hayes, Sands, Tangedal, van Wamelen, Roblot, Solomon and others in a wide variety of cases (see citeDu04). In fact, the conjectures are now used to compute class fields over totally real number fields in the software package Pari. Unfortunately, the conjectures have only been proved in the cases where explicit class field theory is already known (k is either $\mathbb{Q}$ or imaginary quadratic). Nevertheless, the conjecture provides new insight into an old problem.

Chapter 2

The Extended First Order Abelian Stark Question

2.1 Statement of the Question

In a conversation with Dummit in October, 1994, Stark suggested that few computations had been done on Stark's Conjecture testing its "functoriality," i.e., the compatibility of various Stark units when their fields are subfields of a common field. In constructing examples to test this behavior, Dummit encountered some fields in which all the characters were of rank 1 (i.e., the L-functions all had a zero at $s = 0$ of order at least 1), yet there was no single place splitting completely in the field. Over the course of several years, Dummit formulated (but did not publish, although some lectures on the topic were given in 1997-2000) a "robust Stark Conjecture," mentioning this work to Stark in 1998. The feature of a single unit serving as an "L-function evaluator" is lost in Dummit's version.

In preparing an MSRI talk in December 2001, Stark had the idea that this feature could be recovered in his extended version of the Conjecture, still forgoing the assumption of a totally split prime. One of the reasons for the split prime condition in the original conjecture was to guarantee that all the L-functions would vanish at $s = 0$. Throughout, we assume that $|S| \geq 2$ to guarantee that the L-

function for the trivial character vanishes at $s = 0$.

Lemma 2.1. *If S contains a prime v which splits completely, then $L_S(0, \chi) = 0$ for all $\chi \in \widehat{G}$.*

Proof. Note that this is a special case of Lemma 1.2. However, it is worth noting the cause for the vanishing at $s = 0$ of the L-functions here.

If the splitting prime is infinite, then K is totally real. In this case, all the characters in $\widehat{G}$ are even. The functional equation for abelian L-functions of an even character implies that $L_S(s, \chi)$ vanishes at $s = 0$ (see [13, p. 35]).

Suppose the splitting prime v is finite. Then the Frobenius automorphism σ_v is the identity in G, and so $\chi(\sigma_v) = 1$ for all $\chi \in \widehat{G}$. The fact that $L_S(0, \chi) = 0$ follows from Equation 2.1 below:

$$L_S(s, \chi) = \left(1 - \frac{\chi(\sigma_v)}{\mathbf{N}v^s}\right) \cdot L_{S \setminus \{v\}}(s, \chi) \quad \square \tag{2.1}$$

If more than one prime split completely and $|S| \geq 3$, then the order of vanishing of the L-functions would be greater than or equal to two. The First Order Abelian Stark Conjecture holds trivially with $\varepsilon = 1$ in this situation. The case when $|S| = 2$ and both primes split completely is treated in Proposition IV.3.1 in [12]. So we never have to consider when more than one prime splits completely for the first order conjectures.

Stark believed that some version of the conjectures should hold true for any set S for which $L_S(0, \chi) = 0$ for all $\chi \in \widehat{G}$. The Stark unit ε should still have the same algebraic properties: ε is an S-unit which evaluates $L'_S(0, \chi)$ for all χ and $K(\varepsilon^{1/W_K})$ is an abelian extension of k. However, the particulars of the Stark regulator remained unclear.

For the Extended First Order Abelian Stark Question, we replace the condition of the set S containing a distinguished prime which splits completely in K/k with the notion of a 1-covering.

Definition.

1. A *1-covering of* $\widehat{G}$ is a finite set S of primes in k such that for all $\chi \in \widehat{G}$, there is a $v \in S$ such that $\chi\big|_{G_v} = 1$. More generally, a 1-covering of a subset of $\widehat{G}$ is a finite set S of primes in k such there is a $v \in S$ such that $\chi\big|_{G_v} = 1$ for all χ in the subset of $\widehat{G}$. If S contains all ramified and infinite primes of K/k, then S being a 1-covering of $\widehat{G}$ is equivalent to $L_S(0, \chi) = 0$ for all $\chi \in \widehat{G}$.

2. $\widehat{G}_{1,S}$ is the subset of characters χ for which $L_S(s, \chi)$ has precisely first order vanishing at $s = 0$. Equivalently for nontrivial characters, $\chi \in \widehat{G}_{1,S}$ if and only if $\chi\big|_{G_v} = 1$ for precisely one prime $v \in S$.

3. A *1-subcovering of* $\widehat{G}_{1,S}$ is a subset S' of S which is a 1-covering of $\widehat{G}_{1,S}$.

4. The *minimal 1-subcovering of* $\widehat{G}_{1,S}$, denoted as $S_{\min}$, is the set of primes $v \in S$ such that there is a $\chi \in \widehat{G}_{1,S}$ with $\chi\big|_{G_v} = 1$. Equivalently,

$$S_{\min} = \bigcap S'$$

where S' runs through all 1-subcoverings of $\widehat{G}_{1,S}$.

Lemma 2.2.

$$S_{\min} = \bigcap S'$$

where S' runs through all 1-subcoverings of $\widehat{G}$. Thus, we may refer to $S_{\min}$ as the minimal 1-subcovering of $\widehat{G}$.

Proof. $S_{\min} \subseteq \bigcap S'$ is clear, since every 1-subcovering of $\widehat{G}$ is also a 1-subcovering of $\widehat{G}_{1,S}$. If $v \notin S_{\min}$, then for any $S' \subseteq S$ which is a 1-subcovering of $\widehat{G}$, $S' \setminus \{v\}$ is a 1-subcovering of $\widehat{G}_{1,S}$. But then $S' \setminus \{v\}$ is also a 1-covering of $\widehat{G}$, since removing any prime reduces the order of vanishing by at most 1. Hence, $v \notin \bigcap S'$. □

When S contains a single distinguished prime v_0 which splits completely in K/k and $\widehat{G}_{1,S}$ is nonempty, then any 1-subcovering must contain v_0, and $S_{\min} = \{v_0\}$. In general, $S_{\min}$ need not consist of a single prime which splits completely. See Chapter 5 for examples of 1-coverings.

We are now ready to state the Extended First Order Abelian Stark Question.

Question 2.3 (StQ$(K/k,S)$). *Let K/k be an abelian extension of number fields. Let G be the Galois group of K/k and $\widehat{G}$ be the group of characters on G. Let S be a set of primes in k containing all infinite and ramified primes. Suppose that S is a 1-covering for $\widehat{G}$ and $S_{\min}$ is the minimal 1-subcovering of S. Assume that $|S| \geq |S_{\min}| + 1 \geq 2$. For each $v \in S_{\min}$, fix some w lying above v. Does there exist an $\varepsilon \in K^\times$ satisfying the following properties?*

i. $\left|\varepsilon\right|_w = 1$ *for all primes w not lying above a prime in S, i.e., ε is an S-unit. If $S = S_{\min} \cup \{v'\}$, then for w' lying above v', $\left|\varepsilon^\sigma\right|_{w'} = \left|\varepsilon\right|_{w'}$ for all $\sigma \in G$. If $|S| > |S_{\min}| + 1$, then $\left|\varepsilon\right|_w = 1$ for all w not lying above $v \in S_{\min}$.*

ii. *For all $\chi \in \widehat{G}$,*

$$L'_S(0,\chi) = -\frac{1}{W_K}\sum_{\sigma\in G}\chi(\sigma)\log\left(\prod_{v\in S_{\min}}\left|\varepsilon^\sigma\right|_w^{1/|G_v|}\right).$$

iii. $K(\varepsilon^{1/W_K})$ *is an abelian extension of k.*

If StQ$(K/k,S)$ has an affirmative answer, then the Stark unit is unique up to root of unity. In the original conjecture, uniqueness is immediate since one can solve for all the absolute values of ε, which determines ε up to root of unity. One cannot isolate the various absolute values in the same fashion for the above question. Nevertheless, uniqueness of the Stark unit ε has been shown in some unpublished notes of Popescu.

The reason for the assumption that $|S| \geq 2$ is to ensure the L-function for the trivial character vanishes with at least order 1. The reason for the assumption that $|S| \geq |S_{\min}| + 1$ comes from the regulator theory worked out in the unpublished notes of Popescu. There are examples of first order vanishing situations where $S = S_{\min}$, two of which are considered in Chapter 4. Although these examples do not technically satisfy the conditions of StQ$(K/k,S)$, it is interesting to consider whether or not the question continues to have an affirmative answer.

Here are some immediate results from the statement of the general question.

Lemma 2.4. *An affirmative answer to the Extended First Order Abelian Stark Question implies that the First Order Abelian Stark Conjecture is true.*

Proof. When S contains a prime v_0 which splits completely, then $S_{\min} = \{v_0\}$ and G_{v_0} is trivial. Hence, the original conjecture is a special case of the question. □

Lemma 2.5. *The following functorality results hold for the Extended First Order Abelian Stark Question:*

i. $\mathrm{StQ}(K/k, S)$ *has an affirmative answer if all the L-functions for K/k and S have order of vanishing at least two at $s = 0$.*

ii. $\mathrm{StQ}(k/k, S)$ *has an affirmative answer.*

iii. *If $S \subseteq S'$, then an affirmative answer for* $\mathrm{StQ}(K/k, S)$ *implies an affirmative answer for* $\mathrm{StQ}(K/k, S')$.

iv. *If $k \subseteq K' \subseteq K$, then an affirmative answer for* $\mathrm{StQ}(K/k, S)$ *implies an affirmative answer for* $\mathrm{StQ}(K'/k, S)$.

Proof. The proofs are similar to Propositions IV.3.1, 3.2, 3.4, and 3.5 from [12]. In the first two cases, we may take $\varepsilon = 1$ if $|S| \geq 3$; see Lemma 2.6 below when $|S| = 2$. If ε is the Stark unit for $\mathrm{StQ}(K/k, S)$, then the Stark unit for $\mathrm{StQ}(K/k, S \cup \{v\})$ is $\varepsilon^{1-\sigma_v^{-1}}$ for $v \in S' \setminus S$ by Equation 2.1.

For the fourth assertion, let w' be the prime in K' lying between v and w. Let $G' = \mathrm{Gal}(K/K') \subseteq G$. Then $\mathrm{Gal}(K'/k)$ is isomorphic to G/G'. Denote $G_{w|v}$, $G_{w|w'}$, and $G_{w'|v}$ as the relative decomposition groups of the three primes in their respective Galois groups. Note that $|G_{w|v}| = |G_{w|w'}| \cdot |G_{w'|v}|$. By Lemma 1.1, $\left|\alpha\right|_w^{1/|G_{w|v}|} = \left|\alpha\right|_{w'}^{1/|G_{w'|v}|}$ for any $\alpha \in K'$.

Let ε be the Stark unit for $\mathrm{StQ}(K/k, S)$. Choose any $\chi \in \widehat{G}$ such that $\chi\big|_{G'} = 1$, i.e., $\chi \in \widehat{(G/G')}$. Denote $\mathbf{N} = \mathbf{N}_{K/K'}$. Then

$$\begin{aligned}
L_S'(0,\chi) &= -\frac{1}{W_K} \sum_{\sigma \in G} \chi(\sigma) \log\Bigg[\prod_{v \in S_{\min}} \left| \varepsilon^{\sigma} \right|_w^{1/|G_{w|v}|} \Bigg] \\
&= -\frac{1}{W_K} \sum_{\tau \in G/G'} \chi(\tau) \log\Bigg[\prod_{v \in S_{\min}} \Bigg(\prod_{\sigma \in G'} \left| \varepsilon^{\sigma\tau} \right|_w^{1/|G_{w|v}|} \Bigg) \Bigg] \\
&= -\frac{1}{W_K} \sum_{\tau \in G/G'} \chi(\tau) \log\Bigg[\prod_{v \in S_{\min}} \left| (\mathbf{N}\varepsilon)^{\tau} \right|_w^{1/|G_{w|v}|} \Bigg] \\
&= -\frac{1}{W_{K'}} \sum_{\tau \in G/G'} \chi(\tau) \log\Bigg[\prod_{v \in S_{\min}} \left| \left(\mathbf{N}\varepsilon^{W_{K'}/W_K}\right)^{\tau} \right|_{w'}^{1/|G_{w'|v}|} \Bigg]. \qquad (2.2)
\end{aligned}$$

Note that the minimal 1-subcovering of $\widehat{G}$ and $\widehat{(G/G')}$ may be different (although the latter is always a subset of the former). Denote $S'_{\min}$ as the minimal 1-subcovering of $\widehat{(G/G')}$. We may think of $\widehat{(G/G')}$ as the subgroup of characters in $\widehat{G}$ which contain G' in the kernel. If v is in $S_{\min}$ but not $S'_{\min}$, then the characters $\chi \in \widehat{G}_{1,S}$ such that $\chi\big|_{G_v} = 1$ are not in $\widehat{(G/G')}$. Thus, the character sum for $\chi \in \widehat{(G/G')}$

$$\sum_{\tau \in G/G'} \chi(\tau) \log \left| (\mathbf{N}\varepsilon)^{\tau} \right|_{w'}$$

must equal zero, since the original sum at the prime w does not contribute to the value of $L_S'(0,\chi)$. Hence, we may restrict the product from $S_{\min}$ to $S'_{\min}$.

Tate shows that there exists an $\varepsilon' \in K'$ such that $(\varepsilon')^{W_K/W_{K'}} = \zeta \cdot \mathbf{N}\varepsilon$ for some root of unity $\zeta \in K'$ (see [12, IV.3.5] or [2, 6.2.5]). So Equation 2.2 becomes

$$L_S'(0,\chi) = -\frac{1}{W_{K'}} \sum_{\tau \in G/G'} \chi(\tau) \log\Bigg(\prod_{v \in S'_{\min}} \left| (\varepsilon')^{\tau} \right|_{w'}^{1/|G_{w'|v}|} \Bigg).$$

Thus, the Stark unit for $\mathrm{StQ}(K'/k,S)$ is ε'. The abelian condition for $\mathrm{StQ}(K'/k,S)$ follows from the abelian condition for $\mathrm{StQ}(K/k,S)$ and Proposition IV.1.2 from [12]. □

Lemma 2.6. *If $|S| = 2$, then* $\mathrm{StQ}(K/k,S)$ *and* $\mathrm{St}(K/k,S)$ *are equivalent. In particular,* $\mathrm{StQ}(K/k,S)$ *has an affirmative answer when $|S| = 2$.*

Proof. Suppose S satisfies the hypotheses of $\mathrm{StQ}(K/k, S)$. It suffices to show that S contains a prime which splits completely. If S contains a complex infinite prime, then by definition this prime splits completely in K/k. So we need to consider the following remaining cases.

Case 1: $k = \mathbb{Q}$. Over the rationals, any nontrivial extension must contain at least one ramified prime. So S must consist of one ramified prime and the one infinite prime. Hence, the infinite prime must split completely for S to be a 1-covering of $\widehat{G}$.

Case 2: k is real quadratic. Here, S must consist of the two infinite primes and no finite primes. Hence, K must be a subfield of the narrow Hilbert class field. If K has any real embeddings, then the real prime lying below it in k must split completely. If K has only complex embeddings, then $\widehat{G}$ contains a character whose kernel does not contain either decomposition group of the two real infinite primes, contradicting S being a 1-covering.

By Proposition IV.3.10 and Corollary IV.6.6 in [12], $\mathrm{St}(K/k, S)$ is true when $|S| = 2$, so $\mathrm{StQ}(K/k, S)$ also has an affirmative answer. □

Lemma 2.7. *When G is cyclic,* $\mathrm{StQ}(K/k, S)$ *and* $\mathrm{St}(K/k, S)$ *are equivalent.*

Proof. Recall that $\widehat{G}$ is isomorphic to G. Let χ_1 be a generator for $\widehat{G}$. Since S is a 1-covering, there exists a $v_1 \in S$ such that $\chi_1\big|_{G_{v_1}} = 1$. Then for any $\chi \in \widehat{G}$, $\chi = \chi_1^k$ for some integer k, and so $\chi\big|_{G_{v_1}} = 1$. There is only one subgroup of G on which every character vanishes, namely the trivial subgroup. This follows from the non-degeneracy of the character group. Hence, G_{v_1} is trivial and v_1 splits completely. □

The previous lemmas show that the only instances different from the original conjectures occur when $|S| \geq 3$ and when G is a noncyclic abelian group. We will henceforth assume that $|S| \geq 3$ and G is a noncyclic abelian group.

2.2 Reduction

Our basic approach has been to assume the First Order Abelian Stark Conjecture is true for certain intermediate fields K' of K/k (including K itself, if S contains a prime which splits completely). The Stark unit for $\mathrm{StQ}(K/k, S)$ then arises from the Stark units from the intermediate fields.

Suppose $v \in S_{\min}$ and $\chi \in \widehat{G}$ is a character such that $\chi\big|_{G_v} = 1$. Let K' be the fixed field of G_v. Then $\mathrm{Gal}(K'/k) \cong G/G_v$ and K' is the maximal subfield of K in which v splits (K' is called the *decomposition field* of v). Fix some w' in K' which lies above v. Since G_v is in the kernel of χ, we may think of χ as a character on G/G_v. By the First Order Abelian Stark Conjecture for K'/k, for all $\chi \in \widehat{(G/G_v)}$, there exists an $\varepsilon_v \in (K')^\times$ independent of χ such that ε_v is a v-unit,

$$L'_S(0,\chi) = -\frac{1}{W_{K'}} \sum_{\sigma \in G/G_v} \chi(\sigma) \log \left|\varepsilon_v^\sigma\right|_{w'} \tag{2.3}$$

and $K(\varepsilon_v^{1/W_{K'}})$ is an abelian extension over k.

Since ε_v is in the fixed field of G_v, $\varepsilon_v^\sigma = \varepsilon_v$ for all $\sigma \in G_v$. Hence, lifting the sum from G/G_v to G has the effect of dividing by a factor of $|G_v|$:

$$L'_S(0,\chi) = -\frac{1}{W_{K'}} \sum_{\sigma \in G} \chi(\sigma) \log \left|\varepsilon_v^\sigma\right|_{w'}^{1/|G_v|}.$$

By Lemma 1.1, lifting the absolute value from K' to K introduces another factor of $|G_v|$ to the denominator:

$$L'_S(0,\chi) = -\frac{1}{W_{K'}} \sum_{\sigma \in G} \chi(\sigma) \log \left|\varepsilon_v^\sigma\right|_{w}^{1/|G_v|^2}$$

Finally, we adjust the denominator outside the sum to be W_K, which is possible since $W_{K'}$ divides W_K:

$$L'_S(0,\chi) = -\frac{1}{W_K} \sum_{\sigma \in G} \chi(\sigma) \log \left|\left(\varepsilon_v^{W_K/W_{K'}}\right)^\sigma\right|_w^{1/|G_v|^2}$$

We need to show this sum vanishes except when $\chi \in \widehat{G}_{1,S}$ and $\chi\big|_{G_v} = 1$. Denote

$$\mathcal{S}_v = \sum_{\sigma \in G} \chi(\sigma) \log \left|\varepsilon_v^\sigma\right|_w.$$

We prove the following consistency conditions:

Lemma 2.8. *Fix some $v \in S_{\min}$ and $\chi \in \widehat{G}$.*

i. *If $\chi\big|_{G_v} \neq 1$, then $\mathcal{S}_v = 0$.*

ii. *If $\chi\big|_{G_v} = 1$ and there exists some other $v' \in S_{\min}$ such that $\chi\big|_{G_{v'}} = 1$, then $\mathcal{S}_v = 0$.*

Proof.

i. In this case, partition the group G into coset of G_v. Let R be a set of coset representatives of G/G_v. Note that $\varepsilon_v \in K'$ implies that $\varepsilon_v^\tau = \varepsilon_v$ for all $\tau \in G_v$. Hence,

$$\begin{aligned}\mathcal{S}_v &= \sum_{\sigma\in R}\sum_{\tau\in G_v} \chi(\sigma\tau)\log\left|\varepsilon_v^\sigma\right|_w \\ &= \sum_{\sigma\in R}\chi(\sigma)\log\left|\varepsilon_v^\sigma\right|_w\left(\sum_{\tau\in G_v}\chi(\tau)\right).\end{aligned}$$

Since χ is nontrivial on G_v, $\sum_{\tau\in G_v}\chi(\tau) = 0$.

ii. Here, χ may be considered as a character of G/G_v. Since there are at least two primes in S for which χ is trivial on their decomposition groups in K (and therefore in K'), the value of $L'_S(0,\chi) = 0$ in Equation 2.3. Since $\mathcal{S}_v$ is simply the lift of the character sum in Equation 2.3, we conclude that $\mathcal{S}_v = 0$.

□

We now have the following criterion for the Extended First Order Abelian Stark Question. In the theorem below, let K^v denote the fixed field of the decomposition group G_v and W_v denote the number of roots of unity in K^v.

Theorem 2.9. *Let K/k be an abelian extension with Galois group G. Let S be a 1-covering of $\widehat{G}$ and $S_{\min}$ be its minimal 1-subcovering. Suppose that $\mathrm{St}(K^v/k, S)$ holds for all $v \in S_{\min}$ with Stark unit ε_v. Suppose further that for each $v \in S_{\min}$, there exists an $\eta_v \in K^\times$ such that $\varepsilon_v = \eta_v^{|G_v|}$ and such that $K(\eta_v^{1/W_v})$ is abelian over k. Then $\mathrm{StQ}(K/k, S)$ has an affirmative answer.*

Proof. Suppose the above conditions are met. Let the Stark unit ε for K/k and S be defined as follows:

$$\varepsilon = \prod_{v \in S_{\min}} \eta_v^{W_K/W_v} \tag{2.4}$$

By construction, ε is an $S_{\min}$-unit, since it only has nontrivial valuation above primes in $S_{\min}$. By Lemma 2.8, the character sums $\mathcal{S}_v$ vanish unless $\chi \in \widehat{G}_{1,S}$ and $\chi\big|_{G_v} = 1$. Finally, η_v is a v-unit implies that $\left|\varepsilon\right|_w = \left|\eta_v^{W_K/W_v}\right|_w$ for the fixed w lying above v. Thus, for any $\chi \in \widehat{G}$,

$$\begin{aligned} L_S'(0,\chi) &= \sum_{v \in S_{\min}} \left(-\frac{1}{W_K} \sum_{\sigma \in G} \chi(\sigma) \log \left| \left(\varepsilon_v^{W_K/W_v}\right)^{\sigma} \right|_w^{1/|G_v|^2} \right) \\ &= \sum_{v \in S_{\min}} \left(-\frac{1}{W_K} \sum_{\sigma \in G} \chi(\sigma) \log \left| \left(\eta_v^{W_K/W_v}\right)^{\sigma} \right|_w^{1/|G_v|} \right) \\ &= -\frac{1}{W_K} \sum_{\sigma \in G} \chi(\sigma) \log \left(\prod_{v \in S_{\min}} \left| \left(\eta_v^{W_K/W_v}\right)^{\sigma} \right|_w^{1/|G_v|} \right) \\ &= -\frac{1}{W_K} \sum_{\sigma \in G} \chi(\sigma) \log \left(\prod_{v \in S_{\min}} \left| \varepsilon^{\sigma} \right|_w^{1/|G_v|} \right). \end{aligned}$$

Since $K(\eta_v^{1/W_v})$ is abelian over k for each $v \in S_{\min}$, $K\left(\prod \eta_v^{1/W_v}\right) = K\left(\varepsilon^{1/W_K}\right)$ is also abelian over k. This follows from the composition of two abelian fields over a common base field is abelian. Therefore, the abelian condition of $\mathrm{StQ}(K/k, S)$ is met. $\square$

In the next chapter, we show that this criterion is met in a number of cases. Our examples are focused on abelian extensions over the rationals, although several results hold more generally.

Chapter 3

Results

3.1 Unramified Case

With Theorem 2.9 from the previous section in hand, we provide an answer to the Extended First Order Abelian Stark Question when $S_{\min}$ consists solely of unramified primes.

Theorem 3.1. *Let K/k be an abelian extension of number fields, G and $\widehat{G}$ be the Galois group and the group of characters, respectively, and S be a 1-covering. If there is a 1-subcovering $S' = \{v_1, \ldots, v_t\}$ of $\widehat{G}$ consisting of only unramified finite primes in K/k, and if $\mathrm{St}(K'/k, S)$ holds true for all $k \subseteq K' \subseteq K$, then $\mathrm{StQ}(K/k, S)$ has an affirmative answer.*

Proof. Denote $S_0 = S \setminus S'$. Fix some $v_i \in S_{\min}$ and some prime w_i lying above v_i. Since v_i is unramified, the decomposition group G_i of v_i is generated by the Frobenius automorphism σ_i for v_i. Let $\mathsf{f}_i = |G_i| = \mathrm{ord}(\sigma_i)$ be the residual degree for w_i over v_i. Then $\mathrm{Gal}(K_i/k) \cong G/G_i$ and v_i splits completely in K_i. Note that S_0 contains all ramified primes of K_i (since v_i ramifies in K'/k implies v_i ramifies in K/k whenever $k \subseteq K' \subseteq K$). Hence, there is a Stark unit $\varepsilon_i \in K_i^\times$ for the field extension K_i/k and for the set $S_i = S_0 \cup \{v_i\}$.

Lemma 3.2. *For any $j \neq i$ and $\chi \in \widehat{(G/G_i)}$, the Stark unit for K_i/k and $S =$*

$S_i \cup \{v_j\}$ *is* $\varepsilon_i^{1-\sigma_j^{-1}}$.

Proof. Adding a prime v_j to S_i has the effect of multiplying $L_{S_i}(s,\chi)$ by the Euler factor $\left(1 - \frac{\chi(\sigma_j)}{(\mathbf{N}v_j)^s}\right)$. Using the product rule and the fact that $L_{S_i}(0,\chi) = 0$,

$$\begin{aligned}
L_{S_i\cup\{v_j\}}(s,\chi) &= \left(1 - \frac{\chi(\sigma_j)}{\mathbf{N}{v_j}^s}\right) \cdot L_{S_i}(s,\chi) \\
L'_{S_i\cup\{v_j\}}(0,\chi) &= \big(1-\chi(\sigma_j)\big) \cdot L'_{S_i}(0,\chi) \\
&= \big(1-\chi(\sigma_j)\big)\left[-\frac{1}{W_{K_i}} \sum_{\sigma\in G_i} \chi(\sigma) \log \left|\varepsilon_i^\sigma\right|_{w_i}\right] \\
&= -\frac{1}{W_{K_i}}\left[\sum_{\sigma\in G_i} \chi(\sigma)\log\left|\varepsilon_i^\sigma\right|_{w_i} - \sum_{\sigma\in G_i} \chi(\sigma\sigma_j)\log\left|\varepsilon_i^\sigma\right|_{w_i}\right] \\
&= -\frac{1}{W_{K_i}}\left[\sum_{\sigma\in G_i} \chi(\sigma)\log\left|\varepsilon_i^\sigma\right|_{w_i} - \sum_{\sigma\in G_i} \chi(\sigma)\log\left|\varepsilon_i^{\sigma\sigma_j^{-1}}\right|_{w_i}\right] \\
&= -\frac{1}{W_{K_i}} \sum_{\sigma\in G_i} \chi(\sigma)\log\left|\left(\varepsilon_i^{1-\sigma_j^{-1}}\right)^\sigma\right|_{w_i}. \quad \square
\end{aligned}$$

Hence, adding v_j to the set S_i has the effect of applying the group ring element $1-\sigma_j^{-1} \in \mathbb{Z}[G/G_i]$ to the Stark unit ε_i. As we run through all $j \neq i$, the Stark unit for K'/k and $S = S_0 \cup \{v_1,\ldots,v_t\}$ is $\varepsilon_i^{\rho_i}$, where

$$\rho_i = \prod_{j\neq i}\left(1-\sigma_j^{-1}\right) \in \mathbb{Z}[G/G_i].$$

We show that ρ_i is divisible by f_i in $\mathbb{Z}[G/G_i]$. Recall that the Artin map defines an map from the finite unramified primes of k onto elements of G, which send a prime v to its Frobenius element σ_v. Hence, it suffices to prove some more general facts about subsets $\mathcal{S}$ of G which are "1-coverings" of $\widehat{G}$.

Lemma 3.3. *Let* $\mathcal{S}$ *be a subset of an abelian finite group* G *such that for any* $\chi \in \widehat{G}$, $\chi(\sigma) = 1$ *for some* $\sigma \in \mathcal{S}$. *Then in* $\mathbb{Z}[G]$,

$$\rho := \prod_{\sigma\in\mathcal{S}}(1-\sigma) = 0.$$

Proof. Write $\rho = \sum_{\sigma \in G} a_\sigma \sigma$ with $a_\sigma \in \mathbb{Z}$. Applying χ to both sides,

$$\chi(\rho) = \sum_{\sigma \in G} a_\sigma \chi(\sigma) = \prod_{\sigma \in \mathcal{S}} \bigl(1 - \chi(\sigma)\bigr).$$

Since $\mathcal{S}$ is 1-covering of $\widehat{G}$, the product vanishes for all $\chi \in \widehat{G}$. Thus, we have a system of linear equations in the coefficients a_σ, all equal to zero. Since the characters of a group are mutually orthogonal to one another, the linear equations are linearly independent. Therefore, each a_σ must be zero. □

Lemma 3.4. *Let G and $\mathcal{S}$ be defined as in the previous lemma. Fix some $\sigma_0 \in \mathcal{S}$. Then*

$$\rho_0 := \prod_{\substack{\sigma \in \mathcal{S} \\ \sigma \neq \sigma_0}} \bigl(1 - \sigma\bigr)$$

is divisible by $\mathbf{N}\sigma_0 = 1 + \sigma_0 + \ldots + \sigma_0^{\operatorname{ord}(\sigma)-1}$ in $\mathbb{Z}[G]$.

Proof. Write $\rho_0 = \sum_{\sigma \in G} b_\sigma \sigma$ with $b_\sigma \in \mathbb{Z}$. Let ρ be defined as in the previous lemma. Then in $\mathbb{Z}[G]$,

$$\begin{aligned} 0 = \rho = \bigl(1 - \sigma_0\bigr)\rho_0 &= \bigl(1 - \sigma_0\bigr) \sum_{\sigma \in G} b_\sigma \sigma \\ &= \sum_{\sigma \in G} \Bigl(b_\sigma \sigma - b_\sigma \sigma \sigma_0\Bigr) = \sum_{\sigma \in G} \Bigl(b_\sigma - b_{\sigma \sigma_0^{-1}}\Bigr)\sigma. \end{aligned}$$

Therefore, $b_\sigma = b_{\sigma\sigma_0^{-1}}$, and so b_σ is constant on cosets of $\langle \sigma_0 \rangle$. Let R_0 be a set of coset representatives of $G/\langle \sigma_0 \rangle$ Partitioning G into cosets of $\langle \sigma_0 \rangle$, we have

$$\rho_0 = \sum_{\sigma \in R_0} b_\sigma \sigma \sum_{j=0}^{\operatorname{ord}(\sigma_0)-1} \sigma_0^j = \Bigl(\sum_{\sigma \in R_0} b_\sigma \sigma\Bigr)\mathbf{N}\sigma_0. \quad \square$$

Corollary 3.5. *In $\mathbb{Z}[G/\langle \sigma_0 \rangle]$, ρ_0 is divisible by* $\operatorname{ord}(\sigma_0)$.

Proof. $\mathbf{N}\sigma_0$ reduces to $\operatorname{ord}(\sigma_0)$ in $\mathbb{Z}[G/\langle \sigma_0 \rangle]$. □

These lemmas apply to our situation by setting $\rho_0 = \rho_i$, $\sigma_0 = \sigma_i$, and $\operatorname{ord}(\sigma_0) = \mathsf{f}_i = |G_i|$. In particular, ρ_i is divisible by f_i in $\mathbb{Z}[G/G_i]$. Hence, the Stark unit $\varepsilon_i^{\rho_i}$ for K_i/k and S is an f_i^{th} power in K_i. Let $\eta_i = \varepsilon_i^{\rho_i/\mathsf{f}_i}$. The abelian condition for η_i follows from the abelian condition for ε_i with K_i/k and $S_0 \cup \{v_i\}$. Therefore, the conditions of Theorem 2.9 are satisfied.

3.2 Cyclotomic Case

We now turn our attention to cyclotomic extensions of the rationals. One of the original examples of the First Order Abelian Stark Conjecture is for $K/k = \mathbb{Q}(\zeta_m)/\mathbb{Q}$. It follows from the classical theorem of Stickelberger, and the Stark units are normalized Gauss sums [11]. The First Order Abelian Stark Conjecture is also known to hold for any intermediate field of $\mathbb{Q}(\zeta_m)/\mathbb{Q}$, which by the Kronecker-Weber theorem implies the First Order Abelian Stark Conjecture holds for all abelian extensions of $\mathbb{Q}$. We want an answer to the Extended First Order Abelian Stark Question for $K/k = \mathbb{Q}(\zeta_m)/\mathbb{Q}$.

Theorem 3.6. *Suppose $S_{\min} = \{v_0, v_1, \ldots, v_t\}$ consists of one real infinite prime v_0 and unramified finite primes v_i for $1 \leq i \leq t$. Denote $G_i = G_{v_i}$, $K_i = K^{G_i}$, $S_0 = (S \setminus S_{\min}) \cup \{v_0\}$ and $S_i = S_0 \cup \{v_i\}$ for $1 \leq i \leq t$. If $\mathrm{St}(K_i/k, S_i)$ holds for each $v_i \in S_{\min}$, then $\mathrm{StQ}(K/k, S)$ has an affirmative answer.*

Proof. Let σ_i be the Frobenius automorphism of v_i and $\mathsf{f}_i = |G_i| = \mathrm{ord}(\sigma_i)$ as usual. Denote $\tau = \sigma_0$ as complex conjugation associated to v_0. By $\mathrm{St}(K_0/k, S_0)$, there exists a ε_0 such that for any character χ such that $\chi(\tau) = 1$,

$$L'_{S_0}(0, \chi) = -\frac{1}{2} \sum_{\sigma \in G/\langle\tau\rangle} \chi(\sigma) \log \left|\varepsilon_0^{\sigma}\right|_{w_0}.$$

In this case, the Stark unit for K_0/k and S is $\varepsilon_0^{\rho_0}$, where $\rho_0 = \prod_{i=1}^{t}(1 - \sigma_i)$. The same argument in Theorem 3.1 shows that ρ_0 is divisible by $|G_0| = 2$. Hence, we let $\eta_0 = \varepsilon_0^{\rho_0/2} \in K_0^{\times}$.

Now fix $v_i \in S_{\min}$ for some $1 \leq i \leq t$. Let ε_i be the Stark unit for $\mathrm{St}(K_i/k, S_i)$. Following the proof of Theorem 3.1, the Stark unit for $\mathrm{St}(K_i/k, S)$ is $\varepsilon_i^{\rho_i}$, where $\rho_i = \prod_{j \neq i} \big(1 - \sigma_j\big)$. From Lemmas 3.3 and 3.4, $\big(1 - \tau\big)\rho_i$ is divisible by f_i in $\mathbb{Z}[G/G_i]$.

Let $H = G/G_i = G/\langle\sigma_i\rangle$. Note that τ cannot be a power of σ_i. Otherwise, $\chi(\sigma_i) = 1$ implies $\chi(\tau) = 1$, which means v_i could not be an element of $S_{\min}$ ($L_S(s, \chi)$ would have a double zero at $s = 0$ for all $\chi \in \widehat{G}$ with $\chi(\sigma_i) = 1$).

Write $\rho_i = \sum\limits_{\sigma \in H} a_\sigma \sigma$. Choose R to be some set of representatives for $H/\langle\tau\rangle$. Then

$$\begin{aligned} \rho_i &= \sum_{\sigma\in R} a_\sigma \sigma + \sum_{\sigma \in R} a_{\sigma\tau}\sigma\tau \\ &= \sum_{\sigma\in R} a_{\tau\sigma}\big(\sigma + \tau\sigma\big) + \sum_{\sigma\in R}\big(a_\sigma - a_{\tau\sigma}\big)\sigma \\ &= \big(1+\tau\big)\sum_{\sigma\in R} a_{\tau\sigma}\sigma + \sum_{\sigma\in R}\big(a_\sigma - a_{\tau\sigma}\big)\sigma. \end{aligned} \tag{3.1}$$

The first sum of Equation 3.1 has a factor of $\big(1+\tau\big)$ in it, and $\varepsilon_i^{1+\tau} = 1$ since ε_i is a unit at v_0. Therefore,

$$\varepsilon_i^{\rho_i} = \varepsilon_i^{\sum_{\sigma\in R}(a_\sigma - a_{\tau\sigma})\sigma}.$$

Applying $\big(1-\tau\big)$ to both sides of Equation 3.1,

$$\begin{aligned} \big(1-\tau\big)\rho_i &= \big(1-\tau^2\big)\sum_{\sigma\in R} a_{\tau\sigma}\sigma + \big(1-\tau\big)\sum_{\sigma\in R}\big(a_\sigma - a_{\tau\sigma}\big)\sigma \\ &= 0 + \sum_{\sigma\in R}\big(a_\sigma - a_{\tau\sigma}\big)\sigma + \sum_{\sigma\in R}\big(a_{\tau\sigma} - a_\sigma\big)\sigma\tau \\ &= \sum_{\sigma\in H}\big(a_\sigma - a_{\tau\sigma}\big)\sigma. \end{aligned}$$

Hence, $a_\sigma - a_{\tau\sigma}$ is divisible by f_i for all σ, and so $\varepsilon_i^{\rho_i}$ is a f_i^{th} power in K_i as desired. The abelian condition of Theorem 2.9 is satisifed by the same argument at the end of Theorem 3.1. □

This improvement upon Theorem 3.1 is necessary for $\mathrm{StQ}(\mathbb{Q}(\zeta_m)/\mathbb{Q}, S)$. As an illustration, Theorem 5.1 shows that if G contains a subgroup isomorphic to $(\mathbb{Z}/6\mathbb{Z})^2 = \langle\sigma_1\rangle \times \langle\sigma_2\rangle$, then $\mathcal{S}_{\min} = \big\{\sigma_1^2, \sigma_1^3\big\} \cup \big\{\sigma_1^j\sigma_2 \;\big|\; 0 \le j \le 5\big\}$ is a minimal 1-subcovering of $\widehat{G}$. One may choose σ_1 such that $\tau = \sigma_1^3$ is the automorphism corresponding to complex conjugation (i.e., the Frobenius automorphism of the real infinite prime of $\mathbb{Q}$). Then the ρ_i corresponding to σ_1^2 is not divisible by $\mathsf{f}_i = 3$ in $\mathbb{Z}[G/G_i]$. Expanding $\rho_i = \prod\limits_{j=0}^{5}\big(1-\sigma_1^j\sigma_2\big)$ shows that in $\mathbb{Z}[G/G_i]$,

$$\rho_i = 1+\sigma_1 - 3\sigma_2 - 3\sigma_1\sigma_2 + 6\sigma_2^2 + 9\sigma_1\sigma_2^2 - 10\sigma_2^3 - 10\sigma_1\sigma_2^3 + 9\sigma_2^4 + 6\sigma_1\sigma_2^4 - 3\sigma_2^5 - 3\sigma_1\sigma_2^5.$$

However, the coefficients which are not divisible by 3 can be grouped together and have a factor of $(1+\tau)$ removed. Since $(1+\tau)$ kills the Stark unit in this case (having complex absolute value equal to 1), these factors disappear when applied to ε_i. When $(1+\tau)$ is factored out from ρ_i, the result is

$$\rho_i = 3\sigma_2^4 - 3\sigma_2^2 + \big(1+\tau\big)\big(1 - 3\sigma_2 + 9\sigma_2^2 - 10\sigma_2^3 + 9\sigma_2^4 - 3\sigma_2^5\big).$$

Thus,

$$\varepsilon_i^{\rho_i/3} = \varepsilon_i^{\sigma_2^4 - \sigma_2^2}$$

is the desired component of the Stark unit coming from K_i.

Theorem 3.7. *Let m be a positive integer which is either odd or divisible by 4. Let S be a 1-covering of $\widehat{G}$ for the cyclotomic extension $\mathbb{Q}(\zeta_m)/\mathbb{Q}$. For each prime p dividing m, write $m = p^a n$ with $(p,n) = 1$, and suppose that there exists a prime factor l of $\phi(p^a)$ which does not divide $\phi(n)$. Then $S_{\min}$ consists entirely of unramified primes and possibly the one real infinite prime of $\mathbb{Q}$. In particular,* $\mathrm{StQ}(\mathbb{Q}(\zeta_m)/\mathbb{Q}, S)$ *has an affirmative answer.*

Proof. The primes which ramify in $\mathbb{Q}(\zeta_m)/\mathbb{Q}$ are precisely the primes p which divide m. It suffices to show that $p \notin S_{\min}$ for all p dividing m. Let l be the prime factor of $\phi(p^a)$. Separate $\widehat{G}$ into two disjoint subsets, X_1 consisting of all characters defined modulo n and X_2 consisting of all characters for which p divides the conductor. It suffices to show that if $S \setminus \{p\}$ is a 1-subcovering of X_2, then it is a 1-subcovering of X_1.

Suppose $S \setminus \{p\}$ is not a 1-subcovering of X_1. Let $\chi_1 \in X_1$ be a character such that $\chi_1(p) = 1$, $\chi_1(q) \neq 1$ for all $q \in S \setminus \{p\}$, and $\chi_1(-1) = -1$ so that χ_1 is not covered by v_∞. Note that χ_1 takes values in the $\phi(n)^{\text{th}}$ roots of unity. Let $\chi_2 \in X_2$ be a character of conductor p^a and order l, where l is the prime factor of $\phi(p^a)$ which does not divide $\phi(n)$. Then χ_2 takes values in the l^{th} roots of unity, and $\chi_2(-1) = 1$ since l must be odd.

Consider the character $\chi = \chi_1 \cdot \chi_2 \in X_2$. For all $q \in S \setminus \{p\}$, $\chi_1(q)$ is a nontrivial $\phi(n)^{\text{th}}$ root of unity and $\chi_2(q)$ is an l^{th} root of unity. Since $\phi(n)$ and l are relatively prime, $\chi(q) \neq 1$ for all $q \in S \setminus \{p\}$. Furthermore, $\chi(-1) = -1$ since χ_1 is odd and χ_2 is even, so χ is not covered by v_∞. Since $\chi(p) = 0$, χ would not be covered by S. This contradicts S being a 1-covering, and thus $p \notin S_{\min}$. We conclude that $S_{\min}$ contains no ramified primes.

Since the First Order Abelian Stark Conjecture is known to be true for all $K \subseteq \mathbb{Q}(\zeta_m)$, $\mathrm{StQ}(\mathbb{Q}(\zeta_m)/\mathbb{Q}, S)$ has an affirmative answer by Theorem 3.1 if $S_{\min}$ consists of only unramified primes and by Theorem 3.6 if $S_{\min}$ contains the one real infinite prime of $\mathbb{Q}$. □

There are 1-coverings of $\widehat{G}$ with ramified primes in $S_{\min}$ when the condition on m in Theorem 3.7 is not satisfied. One such example is $K/k = \mathbb{Q}(\zeta_{20})/\mathbb{Q}$, $S = \{v_\infty, 2, 3, 5, 11\}$, and $S_{\min} = \{v_\infty, 3, 5, 11\}$, which has been verified to have an affirmative answer to the Extended First Order Abelian Stark Question.

3.3 Multiquadratic Case

We now turn our attention to extensions K/k for which the Galois group G is isomorphic to $\left(\mathbb{Z}/2\mathbb{Z}\right)^m$ for some $m \geq 2$. We call K/k a *multiquadratic extension of rank m*. The original Stark conjecture was proven to hold for these extensions under certain conditions in [6, 7] and extended to most multiquadratic extensions in [5].

As before, the character group $\widehat{G}$ is isomorphic to G, so every nontrivial character has order 2. Denote χ_0 as the trivial character and χ_i as the nontrivial characters for $1 \leq i \leq 2^m - 1$. The kernels $G_i = \operatorname{Ker} \chi_i$ have index 2 in G which correspond to the $2^m - 1$ quadratic extensions K_i/k by Galois theory. Therefore, the Stark units for the multiquadratic extension K/k should arise from the Stark units for the various quadratic extensions K_i/k.

Proposition 3.8 ([12],IV.5.4). *Suppose $[K_i : k] = 2$. If S contains two primes which splits completely in K_i/k, let $\eta_i = 1$. If S contains a single split prime v_i in K_i/k, let η_i be a generator of the free part of the v-units of K_i/k. Let*

$M_i = |\operatorname{Coker}(\mathrm{Cl}_k(S) \to \mathrm{Cl}_{K_i}(S))|$. *Then* $\mathrm{St}(K_i/k, S)$ *holds with Stark unit*

$$\varepsilon_i = \eta_i^{M_i \cdot 2^{|S|-3}}.$$

In [5], the factors M_i are shown to be divisible by 2^r, where $r = r_k(S)$ is the 2-rank of the S_{fin}-class group of k. In fact, Theorem 1 from [5] shows that $\mathrm{St}(K/k, S)$ holds for multiquadratic extensions K/k of rank m if $|S| > m + 1 - r$. We use this fact to prove a similar result for $\mathrm{StQ}(K/k, S)$.

Theorem 3.9. *Let K/k be a multiquadratic extension of rank m. Let S be a 1-covering of $\widehat{G}$ and $S_{\min}$ be its minimal 1-subcovering. Assume that $|S| > m+1-r$. Then* $\mathrm{StQ}(K/k, S)$ *has an affirmative answer.*

Proof. Fix some $v \in S_{\min}$. Let K' be the fixed field of G_v and n_v be the rank of K'/k (that is, $[K' : k] = 2^{n_v}$). Note that $|G_v| = [K : K'] = 2^{m-n_v}$. As usual, K' is the maximal intermediate field in which v splits completely.

By assumption, $|S| > n_v + 1 - r$. Applying Theorem 1 from [5], $\mathrm{St}(K'/k, S)$ holds with Stark unit

$$\varepsilon_v = \prod \eta_i^{M_i \cdot 2^{|S|-n_v-2} (W_{K'}/W_{K_i})}$$

where the product is taken over all quadratic subfields K_i contained in K'. In particular, $\varepsilon_v = \eta_v^N$ for some $\eta_v \in K'$, where $N = 2^{|S|-n_v-2+r}$. Again by assumption, $|S| - n_v - 2 + r \geq m - n_v$ and so $|G_v| = 2^{m-n_v}$ divides N. Furthermore, $\eta_i^{1/W_{K_i}}$ lies in an abelian extension of k by 3.8. Therefore, the conditions of Theorem 2.9 are satisfied. $\square$

We now assume that $m = 2$, that is, K/k is a biquadratic extension.

Theorem 3.10. $\mathrm{StQ}(K/k, S)$ *has an affirmative answer for biquadratic extensions. In fact, the assumption $|S| \geq |S_{\min}| + 1$ may be relaxed to include $S = S_{\min}$.*

Proof. If S contains a prime which splits completely in K/k, then the question was answered by Sands in [5, 6, 7]. Otherwise, each prime splits in at most one of the three intermediate quadratic fields K_i. If $|S| \geq 4$, then the question follows from Theorem 3.9. If $|S| = 3$, then each prime in S splits in a different quadratic subfield of K, which implies every prime of S is in $S_{\min}$. This violates the assumption in $\mathrm{StQ}(K/k, S)$ that $|S| \geq |S_{\min}| + 1$.

In fact, we can show further that there are no biquadratic extensions where $|S| = |S_{\min}| = 3$. It suffices to consider when exactly one prime v_i of S splits in each of the three quadratic subfields K_i, k is totally real, and K is totally complex. The number of real infinite primes of k must be less than or equal to 3 under these restrictions, resulting in the following three cases.

Case 1: $k = \mathbb{Q}$.

Here, $S = \{p, q, v_\infty\}$ for some rational primes p and q. Since K is totally complex, two of the K_i are complex quadratic while the third is real quadratic (where v_∞ splits completely). The splitting criterion for primes in quadratic fields can be found in [1]. It remains to consider whether or not 2 is in S.

If $2 \notin S$, then the two complex quadratic subfields are $\mathbb{Q}(\sqrt{-p})$ and $\mathbb{Q}(\sqrt{-q})$. Since 2 is unramified, p and q must be congruent to 3 modulo 4. Then by quadratic reciprocity, either p splits in $\mathbb{Q}(\sqrt{-q})$ or q splits in $\mathbb{Q}(\sqrt{-p})$, but not both. Hence, S cannot be a 1-covering of $\widehat{G}$.

If $2 \in S$, then one of the two complex quadratic subfields is either $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-2})$, and the other is $\mathbb{Q}(\sqrt{-q})$ for an odd prime q. $p = 2$ splitting in $\mathbb{Q}(\sqrt{-q})$ is equivalent to $q \equiv 7 \pmod 8$. On the other hand, q splitting in $\mathbb{Q}(i)$ is equivalent to $q \equiv 1 \pmod 4$, and q splitting in $\mathbb{Q}(\sqrt{-2})$ is equivalent to $q \equiv 1, 3 \pmod 8$. In either case, S cannot be a 1-covering of $\widehat{G}$.

Case 2: k is real quadratic.

Here, $S = \{\mathfrak{p}, v_\infty^{(1)}, v_\infty^{(2)}\}$ for some finite prime $\mathfrak{p}$ of k. Since only primes in S are allowed to ramify, K must be a subextension of the ray class field associated to the modulus $\mathfrak{p} v_\infty^{(1)} v_\infty^{(2)}$.

Denote H as the Hilbert class field of k and H^+ as the narrow Hilbert class field. To have a biquadratic extension over k with precisely one prime splitting in each quadratic subextension, we must have $K \cap H = k$ and $K \cap H^+$ is one of the three quadratic subfields, say K_3. Let K_1 be the quadratic subfield where $v_\infty^{(1)}$ splits and K_2 be the quadratic subfield where $v_\infty^{(2)}$ splits.

Write $K_1 = k(\sqrt{\alpha_1})$ for some $\alpha_1 \in k$. By construction, $\alpha_1^{(1)} > 0$ and $\alpha_1^{(2)} < 0$. Since $\mathfrak{p}$ ramifies in K_1/k, $\mathrm{Disc}(x^2 - \alpha_1) = (4\alpha_1) = \mathfrak{a}^2\mathfrak{p}^a$ for some integral ideal $\mathfrak{a}$ and positive integer a. If $\mathfrak{p}$ does not lie above 2, then $\mathfrak{p}$ is tamely ramified and $a = 1$.

The nontrivial character χ associated to the subfield K_3 is defined for principal ideals (β) as $\chi((\beta)) = \mathrm{sgn}(\mathbf{N}\beta)$. Suppose $\mathfrak{p}$ does not lie above 2. Then by the discriminant formula,

$$\chi(\mathfrak{p}) = \chi(\mathfrak{a}^2\mathfrak{p}) = \chi((4\alpha_1)) = \chi((\alpha_1)) = \mathrm{sgn}(\mathbf{N}\alpha_1) = -1.$$

Thus, $\mathfrak{p}$ does not split in K_3/k, contradicting S being a 1-covering. Now suppose $\mathfrak{p}$ does lie above 2. If the integer a is odd, then $\chi(\mathfrak{p}) = \chi((\alpha_1)) = -1$ as above. If a is even, then (α_1) is the square of an ideal, contradicting the fact that $\chi((\alpha_1)) = -1$. We conclude that $\mathfrak{p}$ does not split in K_3, so S is not a 1-covering of $\widehat{G}$.

Case 3: k is totally real cubic.

Here, $S = \left\{v_\infty^{(1)}, v_\infty^{(2)}, v_\infty^{(3)}\right\}$ consists of the three infinite primes of k. As observed by Dummit and Hayes, the situation where each prime splits in a different intermediate field implies that the class number is even by a theorem of Armitage and Fröhlich [4]. Hence, the extension K/k cannot be a biquadratic extension. We will consider this situation in Section 4.2. □

These results cover many 1-coverings of multiquadratic extensions. The example in Section 4.1 will consider an multiquadratic example over the rationals with $|S| = |S_{\min}|$ for which $\mathrm{StQ}(K/\mathbb{Q}, S)$ has an affirmative answer, but not for the same reasons as in Theorem 3.9. The example in Section 4.2 will demonstrate that the condition $|S| \geq |S_{\min}| + 1$ cannot be relaxed.

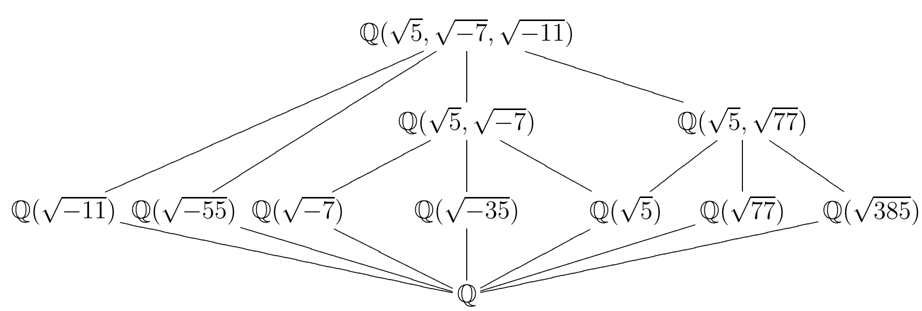
Chapter 4

Examples

The two examples in this chapter explore the veracity of $\mathrm{StQ}(K/k, S)$ when the assumption that $|S| \geq |S_{\min}| + 1$ is replaced with $S = S_{\min}$. The first two conditions of $\mathrm{StQ}(K/k, S)$ appear to hold, but not for the same reasons as the results in Chapter 3. The abelian condition holds in the first example, but appears to fail in the second example.

4.1 Multiquadratic Example

Throughout this section, let $K = \mathbb{Q}(\sqrt{5}, \sqrt{-7}, \sqrt{-11})$. The extension $K/\mathbb{Q}$ is clearly Galois with Galois group isomorphic to $\left(\mathbb{Z}/2\mathbb{Z}\right)^3$. We shall denote $G = \langle \sigma_{100}, \sigma_{010}, \sigma_{001} \rangle$, where $\langle \sigma_{100} \rangle$ fixes $\mathbb{Q}(\sqrt{-7}, \sqrt{-11})$, $\langle \sigma_{010} \rangle$ fixes $\mathbb{Q}(\sqrt{5}, \sqrt{-11})$, and $\langle \sigma_{001} \rangle$ fixes $\mathbb{Q}(\sqrt{5}, \sqrt{-7})$.



Field	Split	Factors	Inert	Ramified
$\mathbb{Q}(\sqrt{-11})$	5	$\left(\frac{3+\sqrt{-11}}{2}\right)$	7	11, v_∞
$\mathbb{Q}(\sqrt{-55})$	7	$\left(7, \frac{1+\sqrt{-55}}{2}\right)$		5, 11, v_∞
$\mathbb{Q}(\sqrt{-7})$	11	$(2+\sqrt{-7})$	5	7, v_∞
$\mathbb{Q}(\sqrt{-35})$	11	$\left(\frac{3+\sqrt{-35}}{2}\right)$		5, 7, v_∞
$\mathbb{Q}(\sqrt{5})$	11, v_∞	$\left(\frac{7+\sqrt{5}}{2}\right)$, w_∞	7	5
$\mathbb{Q}(\sqrt{77})$	v_∞	w_∞	5	7, 11
$\mathbb{Q}(\sqrt{385})$	v_∞	w_∞		5, 7, 11

Table 4.1: Splitting of Primes in Quadratic Subfields

The finite ramified primes of $K/\mathbb{Q}$ are 5, 7, and 11. By considering the Jacobi symbol condition for the splitting of primes in quadratic fields (see [1]), the splitting of these primes in the quadratic subfields of K are listed in Table 4.1.

Let $S = \{5, 7, 11, v_\infty\}$. The kernel of the seven nontrivial characters of $\widehat{G}$ correspond to the seven quadratic subfields of K. Since there is at least one prime in S which splits in each quadratic fields, all the L-functions vanish with at least order one. The L-function for the trivial character also vanishes (with order three) since $|S| = 4$. This follows from Lemma 1.2. Hence, S is a 1-covering of $\widehat{G}$. Furthermore, each prime in S is responsible for the vanishing of at least one L-function. Hence, $S_{\min} = \{5, 7, 11, v_\infty\}$ in this situation.

Viewing the characters of $\widehat{G}$ as characters on the quadratic subfields gives a situation where the First Order Abelian Stark Conjecture are known to hold true and where the Stark units can be explicitly computed. The values of the L-functions for quadratic fields are classical results. For real quadratic fields, the value of the primitive L-function for the nontrivial (even) character χ is

$$L(0, \chi) = 0$$
$$L'(0, \chi) = h \log \varepsilon$$

where h is the class number and ε is the fundamental unit of the real quadratic field. For imaginary quadratic fields, the value of the primitive L-function for the nontrivial character χ is

$$L(0,\chi) = \frac{h}{w/2}$$

where h is the class number and w is the number of roots of unity (equal to 2 except for $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$).

To find the values of the imprimitive L-functions, we apply Equation 2.1 and take a derivative in the case of the imaginary quadratic characters. If S contains more than one prime which splits completely in the quadratic field, then $L'_S(0,\chi) = 0$. Otherwise, let t be the number of primes which remain inert and p be the prime which splits completely in the case of imaginary quadratic fields. Then the first derivatives of the imprimitive L-functions evaluated at $s = 0$ are

$$L'_S(0,\chi) = 2^t\, h \log \varepsilon$$

for real quadratic fields and

$$L'_S(0,\chi) = \frac{1}{w/2}\, 2^t\, h \log p$$

for imaginary quadratic fields. From these formulas, the values of the L_S-functions were evaluated. By choosing a particular prime lying above each $p \in S_{\min}$, an explicit expression of each Stark unit from each quadratic field was found, listed in Table 4.2.

The Stark units are all squares, as predicted by the relative quadratic case. None of them, however, are fourth powers in their respective quadratic fields. Thus, we cannot apply the argument given in the proof of Theorem 3.9. Nevertheless, one of the two square roots of the Stark unit does become a square in the top field K. After making the correct choice of sign for each square root, we have the

Field	h	**Prime**	$L'_S(0,\chi)$	**Stark Unit**
$\mathbb{Q}(\sqrt{-11})$	1	$\mathfrak{p}_5 = \left(\frac{3+\sqrt{-11}}{2}\right)$	$2\log 5$	$\varepsilon_5 = \left(\frac{3+\sqrt{-11}}{3-\sqrt{-11}}\right)^2$
$\mathbb{Q}(\sqrt{-55})$	4	$\mathfrak{p}_7 = \left(7, \frac{1+\sqrt{-55}}{2}\right)$	$4\log 7$	$\varepsilon_7 = \left(\frac{39+4\sqrt{-55}}{49}\right)^2$
$\mathbb{Q}(\sqrt{-7})$	1	$\mathfrak{p}_{11} = \left(2+\sqrt{-7}\right)$	$2\log 11$	$\varepsilon_{11,\mathfrak{p}} = \left(\frac{2+\sqrt{-7}}{2-\sqrt{-7}}\right)^2$
$\mathbb{Q}(\sqrt{-35})$	2	$\mathfrak{q}_{11} = \left(\frac{3+\sqrt{-35}}{2}\right)$	$2\log 11$	$\varepsilon_{11,\mathfrak{q}} = \left(\frac{3+\sqrt{-35}}{3-\sqrt{-35}}\right)^2$
$\mathbb{Q}(\sqrt{5})$	1	$\mathfrak{r}_{11} = \left(\frac{7+\sqrt{5}}{2}\right)$, w_∞	0	1
$\mathbb{Q}(\sqrt{77})$	1	w_∞	$2\log\left(\frac{9+\sqrt{77}}{2}\right)$	$\varepsilon_{77} = \left(\frac{9+\sqrt{77}}{2}\right)^2$
$\mathbb{Q}(\sqrt{385})$	2	w_∞	$2\log\left(95831+4884\sqrt{385}\right)$	$\varepsilon_{385} = \left(95831+4884\sqrt{385}\right)^2$

Table 4.2: L-function Derivatives and Stark Units

following factorizations in K.

$$\begin{aligned}
\varepsilon_5 &= \left(\frac{3+\sqrt{-11}}{3-\sqrt{-11}}\right)^2 = \left(\frac{3+\sqrt{-11}}{2\sqrt{5}}\right)^4 \\
\varepsilon_7 &= \left(-\frac{39+4\sqrt{-55}}{49}\right)^2 = \left(\frac{\sqrt{5}-2\sqrt{-11}}{7}\right)^4 \\
\varepsilon_{11,\mathfrak{p}} &= \left(-\frac{2+\sqrt{-7}}{2-\sqrt{-7}}\right)^2 = \left(\frac{2+\sqrt{-7}}{\sqrt{-11}}\right)^4 \\
\varepsilon_{11,\mathfrak{q}} &= \left(-\frac{3+\sqrt{-35}}{3-\sqrt{-35}}\right)^2 = \left(\frac{3+\sqrt{-35}}{2\sqrt{-11}}\right)^4 \\
\varepsilon_{77} &= \left(-\frac{9+\sqrt{77}}{2}\right)^2 = \left(\frac{\sqrt{-7}+\sqrt{-11}}{2}\right)^4 \\
\varepsilon_{385} &= \left(-\left(95831+4884\sqrt{385}\right)\right)^2 = \left(66\sqrt{-11}+37\sqrt{-35}\right)^4
\end{aligned}$$

When products of the units from different fields are taken, the products become fourth powers in the compositum of the respective quadratic fields, not just in the top field K.

$$\begin{aligned}
\varepsilon_5\cdot\varepsilon_7 &= \left(\frac{3+\sqrt{-11}}{3-\sqrt{-11}}\right)^2\left(\frac{39+4\sqrt{-55}}{49}\right)^2 = \left(\frac{(3+\sqrt{-11})(5-2\sqrt{-55})}{2\cdot5\cdot7}\right)^4 \\
\varepsilon_{11,\mathfrak{p}}\cdot\varepsilon_{11,\mathfrak{q}} &= \left(\frac{2+\sqrt{-7}}{2-\sqrt{-7}}\right)^2\left(\frac{3+\sqrt{-35}}{3-\sqrt{-35}}\right)^2 = \left(\frac{(2+\sqrt{-7})(3+\sqrt{-35})}{2\cdot11}\right)^4 \\
\varepsilon_{77}\cdot\varepsilon_{385} &= \left(\frac{9+\sqrt{77}}{2}\right)^2\left(95831+4884\sqrt{385}\right)^2 = \left(363+\tfrac{259}{2}\sqrt{5}+33\sqrt{77}+\tfrac{37}{2}\sqrt{385}\right)^4
\end{aligned}$$

Hence, the Stark unit for $\mathrm{StQ}(K/\mathbb{Q},S)$ is the fourth root of the product of all the Stark units of the quadratic subfields.

$$\varepsilon = \tfrac{(3+\sqrt{-11})(5-2\sqrt{-55})(2+\sqrt{-7})(3+\sqrt{-35})}{4\cdot 5\cdot 7\cdot 11} \cdot \left(363+\tfrac{259}{2}\sqrt{5}+33\sqrt{77}+\tfrac{37}{2}\sqrt{385}\right)$$

We must now check the abelian condition of $\mathrm{StQ}(K/\mathbb{Q},S)$. We check the equivalent condition due to Coates.

Proposition 4.1 ([12], IV.1.2). *For each $\sigma \in G$, choose n_σ such that $\zeta^\sigma = \zeta^{n_\sigma}$ for each $\zeta \in \mu(K)$. Then $K(\varepsilon^{1/W_K})/k$ is abelian if and only if for each $\sigma \in G$, there exists $\alpha_\sigma \in K$ such that*

$$\varepsilon^{\sigma - n_\sigma} = \alpha_\sigma^{W_K} \quad \text{and} \quad \alpha_\sigma^{\tau - n_\tau} = \alpha_\tau^{\sigma - n_\sigma}$$

for all $\sigma, \tau \in G$.

In the case when $W_K = 2$, we may choose $n_\sigma = 1$ for all σ. In fact, it is enough to check the above condition for a set of generators of G. So it suffices to show that $\varepsilon^{\sigma_{001}-1} = \alpha_{001}^2$, $\varepsilon^{\sigma_{010}-1} = \alpha_{010}^2$, and $\varepsilon^{\sigma_{100}-1} = \alpha_{100}^2$ for some $\alpha_{001}, \alpha_{010}, \alpha_{100} \in K$, and that $\alpha_{001}^{\sigma_{010}-1} = \alpha_{010}^{\sigma_{001}-1}$, $\alpha_{001}^{\sigma_{100}-1} = \alpha_{100}^{\sigma_{001}-1}$, and $\alpha_{010}^{\sigma_{100}-1} = \alpha_{100}^{\sigma_{010}-1}$. This indeed turns out to be the case.

$$\begin{aligned}
\varepsilon^{\sigma_{001}-1} &= \left[\tfrac{(3-\sqrt{-11})(5+2\sqrt{-55})}{2\cdot 5\cdot 7}\left(363+\tfrac{259}{2}\sqrt{5}-33\sqrt{77}-\tfrac{37}{2}\sqrt{385}\right)\right]^2 = \alpha_{001}^2 \\
\varepsilon^{\sigma_{010}-1} &= \left[\tfrac{(2-\sqrt{-11})(3-\sqrt{-35})}{2\cdot 11}\left(363+\tfrac{259}{2}\sqrt{5}-33\sqrt{77}-\tfrac{37}{2}\sqrt{385}\right)\right]^2 = \alpha_{010}^2 \\
\varepsilon^{\sigma_{100}-1} &= \left[\tfrac{(5+2\sqrt{-55})(3-\sqrt{-35})}{2\cdot 7\sqrt{-55}}\left(66\sqrt{-11}-37\sqrt{-35}\right)\right]^2 = \alpha_{100}^2
\end{aligned}$$

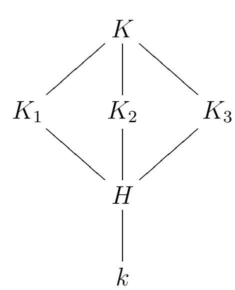
$$\begin{aligned}
\alpha_{001}^{\sigma_{010}-1} &= \left(363+\tfrac{259}{2}\sqrt{5}-33\sqrt{77}-\tfrac{37}{2}\sqrt{385}\right)^2 = \alpha_{010}^{\sigma_{001}-1} \\
\alpha_{001}^{\sigma_{100}-1} &= \left(\tfrac{5-2\sqrt{-55}}{5+2\sqrt{-55}}\right)\left(95831+4884\sqrt{385}\right) = \alpha_{100}^{\sigma_{001}-1} \\
\alpha_{010}^{\sigma_{100}-1} &= \left(\tfrac{3+\sqrt{-35}}{3-\sqrt{-35}}\right)\left(95831+4884\sqrt{385}\right) = \alpha_{100}^{\sigma_{010}-1}
\end{aligned}$$

We conclude that this example satisfies all three conditions of the Extended First Order Abelian Stark Question. It is important to note that this example does not satisfy the conditions of Theorem 3.9. In the proof, the Stark units lifted from the intermediate fields already had the extra powers of two needed within their respective fields. In this example, the units had to be considered as elements of the top field K before the necessary powers appeared. A general proof for multiquadratic extensions that includes the cases when $S = S_{\min}$ would have to consider the index of the S-units in the quadratic fields with respect to the S-units of K, even to find a Stark unit which satisfies the first two conditions of the question.

4.2 Totally Real Cubic Example

A certain class of totally real cubic base fields are of particular interest for the Extended First Order Abelian Stark Question. They were first considered by Dummit and Hayes in [4] for the $\mathfrak{p}$-adic refinement of the First Order Abelian Stark Conjecture for $\mathfrak{p}$ archimedean.

The totally real cubic fields of interest are those with a totally positive system of fundamental units. When this is the case, there are three quadratic intermediate fields K_1, K_2, and K_3 between the Hilbert class field H and the narrow Hilbert class field $K = H^+$. Each of the three real infinite primes of k splits in a different K_i with the other two ramifying. The result is that each K_i/k satisfy the conditions of the First Order Abelian Stark Conjecture with distinguished splitting prime $v_\infty^{(i)}$, while K/k satisfies the conditions of the Extended First Order Abelian Stark Question with $S = S_{\min} = S_\infty = \left\{v_\infty^{(1)}, v_\infty^{(2)}, v_\infty^{(3)}\right\}$.



Dummit and Hayes showed that Gross's local conjecture implies that each of the Stark units ε_i for K_i/k and S_∞ will be a square of some $\eta_i \in K_i$ [4]. In particular, the abelian condition of $\mathrm{St}(K_i/k, S_\infty)$ is trivially satisfied, since $K_i(\sqrt{\varepsilon_i}) = K_i$. By the same techniques as before, the Stark unit for $\mathrm{StQ}(K/k, S_\infty)$ would be the product of the square roots of the Stark units for K_i/k:

$$\varepsilon = \eta_1 \cdot \eta_2 \cdot \eta_3$$

As a numerical test of the Extended First Order Abelian Stark Question in this situation, we consider the example when $k = \mathbb{Q}(\alpha)$, where $\alpha^3 - 22\alpha - 25 = 0$. This example was first calculated separately by Dummit and Tangedal. The field k is a totally real cubic field of discriminant 25717. Let $S = \left\{v_\infty^{(1)}, v_\infty^{(2)}, v_\infty^{(3)}\right\}$ be the three real infinite primes of k corresponding to the embeddings $\alpha^{(1)} = 5.17945\ldots$, $\alpha^{(2)} = -1.21862\ldots$, and $\alpha^{(3)} = -3.96083\ldots$. The fundamental units of k are $\alpha+4$ and $2\alpha^2 + 9\alpha + 8$.

The derivatives of the L-functions were evaluated at $s = 0$ to a precision of 35 digits in Pari/GP. To convert these values into decimal approximations of the Stark units, we first change these into derivatives of the *imprimitive partial zeta functions*, defined as

$$\zeta_S(s, \sigma) = \sum_{\substack{(\mathfrak{a}, S)=1 \\ \sigma_{\mathfrak{a}}=\sigma}} \frac{1}{(\mathbf{N}\mathfrak{a})^s}.$$

By definition,

$$L_S(s, \chi) = \sum_{\sigma \in G} \chi(\sigma) \zeta_S(s, \sigma).$$

By the Fourier inversion formula,

$$\zeta_S(s, \sigma) = \frac{1}{[K : k]} \sum_{\chi \in \widehat{G}} \overline{\chi}(\sigma) L_S(s, \sigma).$$

Hence, an alternative formulation of the second condition of the First Order Abelian Stark Conjecture is

$$\zeta_S'(s, \sigma) = -\frac{1}{W_K} \log \left|\varepsilon^\sigma\right|_w.$$

For each $v_\infty^{(i)}$, fix a $w_\infty^{(i)}$ in K_i lying above $v_\infty^{(i)}$. Denote $\alpha^{(i)}$ as the embedding of $\alpha \in K_i$ corresponding to $w_\infty^{(i)}$. Then the embeddings of the conjugates of ε_i are given by

$$\left(\varepsilon_i^\sigma\right)^{(i)} = e^{-2\zeta_S'(0,\sigma)}.$$

We are interested in finding the minimal polynomials for the square roots $\eta_i = \sqrt{\varepsilon_i}$. By hypothesis, $\eta_i \in K_i$ will have nontrivial valuation at the infinite primes of K_i lying above $v_\infty^{(i)}$ in k and trivial valuation at primes lying above the other two primes of k.

Since one conjugate of η_i is always η_i^{-1}, each η_i will satisfy

$$x^4 - s_1 x^3 + s_2 x^2 - s_1 x + 1 = 0$$

where each of s_1 and s_2 satisfy

$$s_j = a_j + b_j\alpha + c_j\alpha^2$$

for some a_j, b_j, and c_j in $\mathbb{Q}$. Since $\mathcal{O}_k = \mathbb{Z}[\alpha]$ in this case, the coefficients will be integral. Using a naive search method, we found all the coefficients for each η_i and s_j:

η_1	$s_1^{(1)} = 350.5237002408138031\ldots$	$s_1 = 29 + 31\,\alpha + 6\,\alpha^2$
	$s_2^{(1)} = 13392.1630171735303817\ldots$	$s_2 = 1106 + 1186\,\alpha + 229\,\alpha^2$
η_2	$s_1^{(2)} = 19.2963349139655588\ldots$	$s_1 = 22 + \alpha - \alpha^2$
	$s_2^{(2)} = 90.2630515494901634\ldots$	$s_2 = 105 + 6\,\alpha - 5\,\alpha^2$
η_3	$s_1^{(3)} = 50.0630538651472979\ldots$	$s_1 = -13 - 8\,\alpha + 2\,\alpha^2$
	$s_2^{(3)} = 127.6576346628682447\ldots$	$s_2 = -30 - 20\,\alpha + 5\,\alpha^2$

From this, the minimal polynomials of η_i over $\mathbb{Q}$ were computed. Below are the minimal polynomials $f_i(x)$ for each η_i.

$$
\begin{aligned}
f_1(x) = x^{12} &- 351\,x^{11} + 13561\,x^{10} - 7390\,x^9 + 25694\,x^8 - 13553\,x^7 \\
&+ 39701\,x^6 - 13553\,x^5 + 25694\,x^4 - 7390\,x^3 + 13561\,x^2 - 351\,x + 1 \\
f_2(x) = x^{12} &- 22\,x^{11} + 148\,x^{10} - 379\,x^9 + 724\,x^8 - 1039\,x^7 \\
&+ 1150\,x^6 - 1039\,x^5 + 724\,x^4 - 379\,x^3 + 148\,x^2 - 22\,x + 1 \\
f_3(x) = x^{12} &- 49\,x^{11} + 77\,x^{10} - 40\,x^9 + 147\,x^8 + 39\,x^7 \\
&+ 179\,x^6 + 39\,x^5 + 147\,x^4 - 40\,x^3 + 77\,x^2 - 49\,x + 1
\end{aligned}
$$

A particular choice of $w_\infty^{(i)}$ was chosen such that $\left|\eta_i^{\sigma_i}\right|_{w_\infty^{(i)}} = e^{-\zeta'(0,\sigma_0)}$ where σ_0 is the identity of G. For this choice, the minimal polynomial of the conjectural Stark unit ε for K/k is

$$
\begin{aligned}
x^{24} &- 537\,x^{23} + 94631\,x^{22} + 253897\,x^{21} - 81061221\,x^{20} - 3045309741\,x^{19} \\
&+ 382553281913\,x^{18} + 3970254584876\,x^{17} - 16085558174013\,x^{16} \\
&- 2830631362833\,x^{15} + 3078399084807090\,x^{14} - 10593922360433646\,x^{13} \\
&+ 16551004636473558\,x^{12} - 3901253166325746\,x^{11} + 1805016930247669\,x^{10} \\
&+ 490737556448614\,x^9 - 2068051901019\,x^8 - 2784030674202\,x^7 \\
&+ 323638381748\,x^6 + 11282397818\,x^5 + 146274165\,x^4 + 5911288\,x^3 \\
&+ 126941\,x^2 + 623\,x + 1
\end{aligned}
$$

The roots were verified to agree with the values coming from the partial zeta functions to a precision of 35 digits.

However, the abelian condition does not appear to hold. Again, we appealed to the equivalent condition in Proposition 4.1. After finding the conjugates of the Stark unit ε in K/k, the quotients $\varepsilon^{\sigma-1}$ were tested to see whether they were squares in K.

The square testing was performed by two different techniques. The first technique, performed originally by Brett Tangedal, was to attempt factoring $x^2-(\varepsilon^{\sigma-1})$ for each of generator of the Galois group. In this example, Pari found that polynomials factored in K for the three automorphisms of order two, but not for the four automorphisms of order four.

Although the factoring method is a well tested algorithm in Pari, there is the possibility that a negative result could be the result of limitations of the algorithm. Therefore, a second test was desired to reaffirm these results. Under a suggestion by Joe Buhler, the image of each $\varepsilon^{\sigma-1}$ in the residue field was computed for a number of primes. Since each component of ε is a unit in $\mathcal{O}_K$, it would suffice to show that $\varepsilon^{\sigma-1}$ is not a square modulo some prime $\mathfrak{P}$ in K.

After generating a list of how the first 100 rational primes split in k and up to K, two first primes over k which split completely in K were found. (The reason for these choices of primes was simplicity. The size of the residue fields $\mathcal{O}_K/\mathfrak{P}$ equals $\mathbf{N}\mathfrak{P} = p^{\mathsf{f}}$, where p is the rational prime which $\mathfrak{P}$ lies above and f is the residual degree. Choosing f as small as possible simplifies the calculation.)

For this example, the rational primes were 263 and 503. Since both were congruent to 3 modulo 4, it sufficed to test whether $\varepsilon^{\sigma-1}$ is an odd power of a generator of $\mathcal{O}_K/\mathfrak{P}$ to demonstrate that $\varepsilon^{\sigma-1}$ was not a square. Using the "ideallog" function in Pari, it was shown that for each $\sigma \in G$ of order four, $\varepsilon^{\sigma-1}$ was an odd power of a generator for the eight first degree primes in K above 503.

With both of these tests in hand, we can be quite confident that the abelian condition of the Extended First Order Abelian Stark Question is not satisfied for this example. The same result has been verified for four other totally real cubic fields in the same category. It would be possible to verify these results for all 29 totally real cubic fields with discriminant less than 150000 which have totally positive system of fundamental units. However, limitations of the naive search algorithm utilized and the length of time required to perform the calculations for each field (which have not been optimized for speed) prevents us from going further. At this point, it would worthwhile to find a reason for the apparent lack of success in the abelian condition.

It is important to note that these cubic examples do not satisfy the condition that $|S| \geq |S_{\min}| + 1$. When an additional prime v is added to S (finite and unramified by necessity), the abelian condition for the modified Stark unit $\varepsilon^{1-\sigma_v^{-1}}$ is satisfied. This gives us reason to believe that requiring at least one more prime in S than in $S_{\min}$ is a necessary assumption. It may be possible to replace the condition with a weaker condition when $S = S_{\min}$. Dummit has proposed such a replacement (the “robust” Stark Conjecture in [3]), but it is currently unpublished and unknown to the author.

Chapter 5

Examples of 1-Coverings

We now present some interesting examples of 1-coverings. By Chebatorev Density Theorem, there are infinitely many unramified primes which have a given element of G as its Frobenius automorphism. Hence, we may specify a subset of G as a 1-covering without referring to specific primes in k. We will denote subsets of G as $\mathcal{S}$.

Theorem 5.1. *Let p be a prime positive integer and n be a composite positive integer.*

i. *If G contains a subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$ generated by σ_1 and σ_2, then*

$$\mathcal{S} = \left\{\sigma_1\right\} \cup \left\{\sigma_1^j \sigma_2 \;\middle|\; 0 \leq j \leq p-1\right\}$$

is a minimal 1-subcovering of $\widehat{G}$.

ii. *If G contains a subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^p$ generated by $\sigma_1, \sigma_2, \ldots, \sigma_p$, then*

$$\mathcal{S} = \left\{\sigma_j \;\middle|\; 1 \leq j \leq p\right\} \cup \left\{\sigma_j \sigma_{j+1} \;\middle|\; 1 \leq j \leq p-1\right\} \cup \cdots \cup \left\{\sigma_1 \sigma_2 \cdots \sigma_p\right\}$$

is a minimal 1-subcovering of $\widehat{G}$, where $\mathcal{S}$ consists of all products of consecutive indices of length less than or equal to p.

iii. *If G contains a subgroup isomorphic to $(\mathbb{Z}/n\mathbb{Z})^2$ generated by σ_1 and σ_2, then*

$$\mathcal{S} = \left\{\sigma_1^{n/p} \;\middle|\; p|n, p \text{ prime}\right\} \cup \left\{\sigma_1^j\sigma_2 \;\middle|\; 0 \leq j \leq n-1\right\}$$

is a minimal 1-subcovering of $\widehat{G}$.

Proof. When $S_{\min}$ consists of unramified primes, it suffices to show that for each $\chi \in \widehat{G}$, there exists a $\sigma \in \mathcal{S}$ such that $\chi(\sigma) = 1$.

i. Choose any $\chi \in \widehat{G}$. If either $\chi(\sigma_1) = 1$ or $\chi(\sigma_2) = 1$, then there is nothing to show. Otherwise, $\chi(\sigma_1)$ and $\chi(\sigma_2)$ are primitive p^{th} roots of unity, in which case exactly one of $\chi(\sigma_1^j\sigma_2) = \chi(\sigma_1)^2\chi(\sigma)$ must equal 1.
Minimality follows from the ability to construct characters on $\widehat{G}$ which are equal to 1 on exactly one element of $\mathcal{S}$. For example, there exists a χ such that $\chi(\sigma_1) = 1$ and $\chi(\sigma_2) = \zeta_p$, which shows that σ_1 is in $\mathcal{S}_{\min}$. This can be performed likewise for each element of $\mathcal{S}_{\min}$.

ii. Choose any $\chi \in \widehat{G}$. If any one of $\chi(\sigma_1) = 1$, $\chi(\sigma_1\sigma_2) = 1$, ..., $\chi(\sigma_1 \cdots \sigma_p) = 1$, then there is nothing to show. If none are equal to 1, then by the pigeonhole principle, there exists two equal to the same p^{th} root of unity, say

$$\chi(\sigma_1 \cdots \sigma_i) = \chi(\sigma_1 \cdots \sigma_j)$$

for some $1 \leq i < j \leq p$. Then $\chi(\sigma_{i+1} \cdots \sigma_j) = 1$, which by definition is an element of $\mathcal{S}$.
Once again, minimality follows from the ability to construct a character with any given p^{th} roots of unity. In particular, for any given $\sigma_{i+1} \cdots \sigma_j \in \mathcal{S}$, define $\chi \in \widehat{G}$ by $\chi(\sigma_1 \cdots \sigma_i) = \chi(\sigma_1 \cdots \sigma_j) = \zeta_p$ and let $\chi(\sigma_1 \cdots \sigma_k)$ run through all the other distinct primitive p^{th} roots of unity for $k \neq i$, $k \neq j$. Then χ is equal to 1 on $\sigma_{i+1} \cdots \sigma_j$ and no other elements of $\mathcal{S}$, hence establishing minimality of $\mathcal{S}$.

iii. Choose any $\chi \in \widehat{G}$. If $\chi(\sigma_1)$ is not a primitive n^{th} root of unity, then at least one of $\chi(\sigma_1^{n/p})$ must be equal to 1. If $\chi(\sigma_1)$ is a primitive n^{th} root of unity, then precisely one of $\chi(\sigma_1^j\sigma_2)$ will be equal to 1.

Minimality follows from the ability to choose $\chi(\sigma_1)$ to be a primitive $(n/p)^{\text{th}}$ root of unity for any prime p which divides n.

These three examples were important in providing evidence for the unramified and cyclotomic cases. The first set of examples established our basic motivation with $p = 2$. The second set of examples was computed using Mathematica for $p = 5$ and $p = 7$, which provided further evidence of the Extended First Order Abelian Stark Question. In particular, the group ring element ρ from Section 3.1 had hundreds of terms $\mathbb{Z}[G]$ for $p = 5$ and thousands of terms for $p = 7$. Each coefficient was precisely 5 and 7, respectively. The third set of examples with $n = 6$ established the necessity for the calculation in Section 3.2.

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