

Variations on a Theme of Stark

Stefan Erickson

Colorado College

March 29, 2007

Birch Swinnerton-Dyer Conjecture

First nonzero term of L -functions of elliptic curves E .

$$L(s, E) = a_r (s - 1)^r + \dots$$

Weak BSD: $r = \text{Rank}_{\mathbb{Z}}(E(\mathbb{Q}))$

Strong BSD: Explicit formula for a_r in terms of elliptic curve constants.

Stark Conjectures

First nonzero term of Artin L -functions of number fields K/k .

Let ρ be a representation of $\text{Gal}(K/k)$.

$$L(s, \rho) = a_r s^r + \dots$$

Stark: Formula for a_r computed “algebraically.”

Recent work has shown that both conjectures are related through the Equivariant Tamagawa Number Conjecture.

Notation

- K/k is an abelian extension of number fields.
- W_K is the number of roots of unity in K .
- $G = \text{Gal}(K/k)$ is the Galois group of K/k .
- \widehat{G} is the group of characters on G .

- v is a prime in k , w is a prime in K lying above v .
- σ_v is the Frobenius automorphism of v ($\alpha^{\sigma_v} \equiv \alpha^{\mathbf{N}v} \pmod{w}$).
- G_v is the decomposition group of v ($w^{G_v} = w$).
- S is a finite set of primes of k , including all ramified and infinite primes. Throughout, assume $|S| \geq 3$.

v -adic Absolute Values

Finite prime = prime ideal in ring of integers.

Infinite prime = embedding of k into \mathbb{R} or \mathbb{C} .

$$|\alpha|_v = \begin{cases} (\mathbf{N}v)^{-\text{ord}_v(\alpha)}, & \text{if } v \text{ is finite} \\ |\alpha|, & \text{if } v \text{ is real infinite} \\ |\alpha|^2, & \text{if } v \text{ is complex infinite} \end{cases}$$

If w lies over v and $\alpha \in k$, then $|\alpha|_v = |\alpha|_w^{1/|G_v|}$

For $\operatorname{Re}(s) > 1$, the **imprimitive L -function** of χ and S is given by

$$L_S(s, \chi) = \prod_{v \notin S} \left(1 - \frac{\chi(\sigma_v)}{\mathbf{N}v^s} \right)^{-1}$$

This function can be extended holomorphically to the entire complex plane if χ is nontrivial, and meromorphically with a simple pole at $s = 1$ if χ is trivial.

Adding a Prime to S

If v is an unramified finite prime, then

$$L_{S \cup \{v\}}(s, \chi) = \left(1 - \frac{\chi(\sigma_v)}{\mathbf{N}v^s}\right) \cdot L_S(s, \chi)$$

The order of vanishing of $L_S(s, \chi)$ at $s = 0$ increases by one for each $v \in S$ for which $\chi(\sigma_v) = 1$.

If S contains a prime v_0 which splits completely, then σ_{v_0} is the identity of G , and $L_S(0, \chi) = 0$ for all $\chi \in \widehat{G}$.

First Order Abelian Stark Conjecture (1980)

Conjecture ($\text{St}(K/k, S)$). *Suppose S contains a prime v_0 which splits completely. Fix some w_0 lying above v_0 . Then there exists an $\varepsilon \in K^\times$, unique up to root of unity, such that*

i. ε is a v_0 -unit, that is, $|\varepsilon|_w = 1$ for all w not lying above v_0 .

ii. For all $\chi \in \widehat{G}$,

$$L'_S(0, \chi) = -\frac{1}{w_K} \sum_{\sigma \in G} \chi(\sigma) \log |\varepsilon^\sigma|_{w_0}$$

iii. $K(\sqrt[w_K]{\varepsilon})$ is an abelian extension of k .

Examples of Stark Units

1. $K = \text{max real subfield of } \mathbb{Q}(\zeta_m), k = \mathbb{Q}, S = \{p|m\} \cup \{\infty\}.$

Then $\varepsilon = \left(2 \sin \frac{\pi}{m}\right)^2 = (1 - \zeta_m)(1 - \zeta_m^{-1}).$

2. $K = \mathbb{Q}(\zeta_m), k = \mathbb{Q}, S = \{p|m\} \cup \{\infty\} \cup \{p_0 \equiv 1 \pmod{m}\}.$

Then $\varepsilon = \left(\frac{\mathcal{G}(p_0)}{\sqrt{p_0}}\right)^{W_K}$ is a power of a normalized Gauss sum.

Explicit Class Field Theory

The main application of the First Order Stark Conjecture is to construct maximal abelian extensions, conjecturally answering Hilbert's Twelfth Problem in certain cases.

The software package Pari uses Stark units to compute the Hilbert Class Field for totally real number fields.

Dummit's Question

Can we relax the condition that S contains a prime which splits completely?

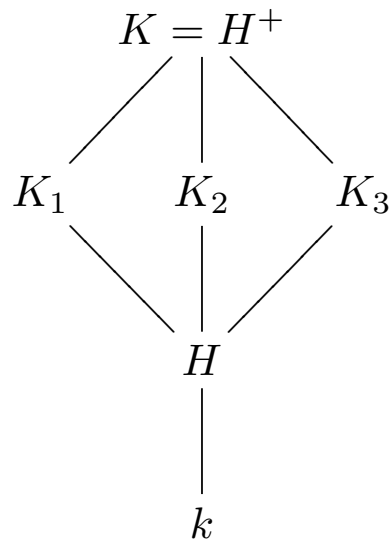
Dummit's Example

k : Totally real cubic field, totally positive system of fund. units.

H : Hilbert class field (all primes are unramified)

K : Narrow Hilbert class field (all *finite* primes are unramified)

K_i : Fixed field of the decomposition group for $v_\infty^{(i)}$.



Replacing an Assumption

The purpose of the splitting prime v_0 is to create a first order zero at $s = 0$ for all $L_S(s, \chi)$.

To get a first order zero conjecture, it is enough to choose any set S such that $L_S(0, \chi) = 0$ for all $\chi \in \widehat{G}$. We call S a **1-covering** of \widehat{G} .

The **minimal 1-subcovering**, denoted by S_{\min} , consists of all $v \in S$ such that for some $\chi \in \widehat{G}$, $\chi|_{G_v} = 1$ and $L_S(s, \chi)$ has precisely a first order zero at $s = 0$. Alternatively,

$$S_{\min} = \bigcap S'$$

where S' runs through all 1-subcoverings of \widehat{G} .

Extended First Order Abelian Stark Conjecture (2001)

Conjecture ($\text{St}'(K/k, S)$). *Let S be a 1-covering of \widehat{G} containing all infinite and ramified primes. Let S_{\min} be the minimal 1-subcovering. Assume that $|S| \geq |S_{\min}| + 1$. For each $v \in S_{\min}$, fix some w lying above v . Then there exists an $\varepsilon \in K^\times$, unique up to root of unity, with the following properties:*

i. $|\varepsilon|_w = 1$ for all w not lying above a prime in S_{\min} .

ii. For all $\chi \in \widehat{G}$,

$$L'_S(0, \chi) = -\frac{1}{W_K} \sum_{\sigma \in G} \chi(\sigma) \log \left(\prod_{v \in S_{\min}} |\varepsilon^\sigma|_w^{1/|G_v|} \right)$$

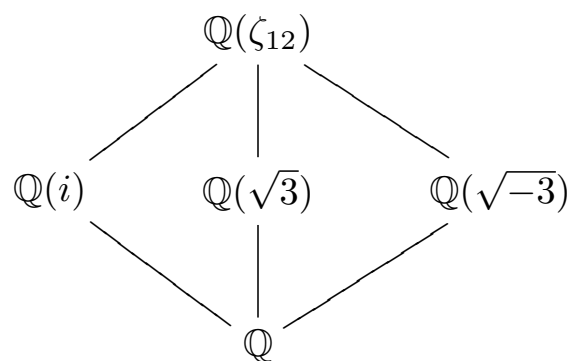
iii. $K(\sqrt[{}^W K]{\varepsilon})$ is an abelian extension of k .

Basic Example

$$K/k = \mathbb{Q}(\zeta_{12})/\mathbb{Q}, G = \text{Gal}(K/k) \cong (\mathbb{Z}/12\mathbb{Z})^\times$$

	1	5	7	11
χ_1	1	1	1	1
χ_2	1	-1	-1	1
χ_3	1	-1	1	-1
χ_4	1	1	-1	-1

$$S = \{\infty, 2, 3, 5, 7\}, S_{\min} = \{\infty, 5, 7\}$$



Stark Units for Intermediate Fields

Field	Prime	Factor	S-Unit	G_p
$\mathbb{Q}(i)$	$p = 5$	$\mathfrak{p}_5 = (2 + i)$	$\varepsilon_5 = \frac{2+i}{2-i}$	$\{1, 5\}$
$\mathbb{Q}(\sqrt{-3})$	$q = 7$	$\mathfrak{q}_7 = \left(\frac{5+\sqrt{-3}}{2}\right)$	$\varepsilon_7 = \frac{5+\sqrt{-3}}{5-\sqrt{-3}}$	$\{1, 7\}$
$\mathbb{Q}(\sqrt{3})$	∞	∞	$\varepsilon_\infty = 2 + \sqrt{3}$	$\{1, 11\}$

∞ splits in $\mathbb{Q}(\sqrt{3})$, $\varepsilon_\infty = (2 - \sqrt{3})^4$ for $\infty_{\mathbb{R}} : (\text{fund. unit}) \mapsto 2 + \sqrt{3}$.

5 splits in $\mathbb{Q}(i)$, $\varepsilon_5 = \left(\frac{2+i}{2-i}\right)^4 = \left(\frac{\mathcal{G}(5)}{\sqrt{5}}\right)^{16}$ for $\mathfrak{p}_5 = (2 + i)$.

7 splits in $\mathbb{Q}(\sqrt{-3})$, $\varepsilon_7 = \zeta_3 \left(\frac{5+\sqrt{-3}}{5-\sqrt{-3}}\right)^4 = \left(\frac{\mathcal{G}(7)}{\sqrt{-7}}\right)^{12}$ for $\mathfrak{q}_7 = \left(\frac{5+\sqrt{3}}{2}\right)$.

Let $\varepsilon = \varepsilon_\infty^3 \cdot \varepsilon_5^{\frac{3}{2}} \cdot \varepsilon_7$. Then for any $\chi \in \widehat{G}$,

$$L'_S(0, \chi) = -\frac{1}{12} \sum_{\sigma \in G} \chi(\sigma) \log \left(|\varepsilon^\sigma|_{\mathfrak{p}_\infty}^{1/2} |\varepsilon^\sigma|_{\mathfrak{p}_5}^{1/2} |\varepsilon^\sigma|_{\mathfrak{p}_7}^{1/2} \right)$$

and $\sqrt[12]{\varepsilon} = (2 - \sqrt{3}) \left(\frac{\mathcal{G}(5)}{\sqrt{5}}\right)^2 \left(\frac{\mathcal{G}(7)}{\sqrt{-7}}\right)$ lies in an abelian extension of \mathbb{Q} .

Results

w	K	Suppose $v \in S_{\min}$ and $\chi _{G_v} = 1$. Let $K' = K^{G_v}$.
w'	K'	From $\text{St}(K'/k, S)$, there is an $\varepsilon_v \in K'$ such that
		$L'_S(0, \chi) = -\frac{1}{W_{K'}} \sum_{\sigma \in G/G_v} \chi(\sigma) \log \varepsilon_v^\sigma _{w'}$
v	k	

Lifting the sum to G and the absolute value to w ,

$$L'_S(0, \chi) = -\frac{1}{W_K} \sum_{\sigma \in G} \chi(\sigma) \log \left| \left(\varepsilon_v^{W_K/W_{K'}} \right)^\sigma \right|_w^{1/|G_v|^2}$$

By a variety of methods, we can show ε_v is a $|G_v|^{\text{th}}$ power in K' .

The Stark unit for K is $\varepsilon = \prod_{v \in S_{\min}} \eta_v^{W_K/W_{K'}}$, where $\eta_v^{|G_v|} = \varepsilon_v$.

Theorem. *If*

- K/k abelian such that $\text{St}(K'/k, S)$ holds for all $k \subseteq K' \subseteq K$
- $S' = \{v_1, v_2, \dots, v_t\}$ is a 1-subcovering consisting of only unramified finite primes

then $\text{St}'(K/k, S)$ is true.

K Let ε_i be the Stark unit for K_i and $S_i = (S \setminus S_{\min}) \cup \{v_i\}$.

Then the Stark unit for $\text{St}'(K/k, S)$ is

K_i

$$\varepsilon = \prod_{i=1}^t \varepsilon_i \frac{W_K}{W_{K_i}} \frac{\rho_i}{|G_i|}$$

k where $\rho_i = \prod_{j \neq i} (1 - \sigma_j^{-1}) \in \mathbb{Z}[G]$ is divisible by $|G_i|$.

Theorem. *$\text{St}'(K/k, S)$ is true when S has a 1-subcovering S' consisting of only unramified finite primes and one infinite prime.*

The theorem follows from the previous theorem, plus an adjustment when v_∞ is in S' .

These theorems imply that $\text{St}'(\mathbb{Q}(\zeta_m)/\mathbb{Q}, S)$ is true for most m .

Theorem. *Let K/k be a multiquadratic extension of rank m , i.e., $G = \text{Gal}(K/k) \cong (\mathbb{Z}/2\mathbb{Z})^m$. Then $\text{St}'(K/k, S)$ is true if $|S| > m + 1 - r$, where r is the 2-rank of the S_{fin} -class group of k .*

Corollary. *$\text{St}'(K/k, S)$ is true for biquadratic extensions.*

When $|S|$ is big enough, results of Tate and Sands give extra powers of two. These powers are usually enough to get a factor of $|G_i|$.

Dummit's Example Revisited

$S = S_{\min} = S_{\infty} = \{v_{\infty}^{(1)}, v_{\infty}^{(2)}, v_{\infty}^{(3)}\}$ is a 1-covering for K/k .

Gross's local refinement of $\text{St}(K_i/k, S_{\infty})$ predicts $\varepsilon_i = \eta_i^2$ in K_i .

$\text{St}(K_i/k, S_{\infty})$ has been computationally verified for 26 base fields by Dummit and Hayes.

$\varepsilon = \prod \eta_i \in K^{\times}$ is the correct L -function evaluator for $\text{St}'(K/k, S_{\infty})$.

The abelian condition of $\text{St}'(K/k, S_{\infty})$ does *not* hold for ε .

Does not satisfy $|S| \geq |S_{\min}| + 1$. Adding a prime to S restores the abelian condition.

Open Questions

- Would like proof for $k = \mathbb{Q}$.
- Extension to r^{th} order vanishing. (Popescu, Emmons)
- Function field analogues.
- p -adic refinements.